

Nonlinear Robust Tracking Control of a Quadrotor UAV on SE(3)

Taeyoung Lee*, Melvin Leok†, and N. Harris McClamroch

Abstract—This paper provides nonlinear tracking control systems for a quadrotor unmanned aerial vehicle (UAV) that are robust to bounded uncertainties. A mathematical model of a quadrotor UAV is defined on the special Euclidean group, and nonlinear output-tracking controllers are developed to follow (1) an attitude command, and (2) a position command for the vehicle center of mass. The controlled system has the desirable properties that the tracking errors are uniformly ultimately bounded, and the size of the ultimate bound can be arbitrarily reduced by control system parameters. Numerical examples illustrating complex maneuvers are provided.

I. INTRODUCTION

A quadrotor unmanned aerial vehicle (UAV) consists of two pairs of counter-rotating rotors and propellers. It has been envisaged for various applications such as surveillance or sensor networks as well as for educational purposes.

Several nonlinear controllers have been developed, by using backstepping [1], sliding mode technique [2], nonlinear H_∞ controller [3], and adaptive neural network [4]. All of these controllers are based on Euler angles, and they exhibit singularities when representing complex rotational maneuvers of a quadrotor UAV, thereby significantly restricting their ability to achieve complex flight maneuvers.

An attitude control system based on quaternions is applied to a quadrotor UAV [5]. Quaternions do not have singularities, but they have ambiguities in representing an attitude, as the three-sphere S^3 double-covers $SO(3)$. As a result, in a quaternion-based attitude control system, convergence to a single attitude implies convergence to either of the two disconnected, antipodal points on S^3 [6]. Therefore, depending on the particular choice of control inputs, a quaternion-based control system may become discontinuous when applied to actual attitude dynamics [7], and it may also exhibit unwinding behavior, where the controller rotates a rigid body through unnecessarily large angles [8].

Attitude control systems also have been developed directly on the special orthogonal group, $SO(3)$ to avoid the singularities associated with Euler-angles and the ambiguity of quaternions [9], [10], [11]. By following this geometric approach, the dynamics of a quadrotor UAV is globally

expressed on the special Euclidean group, $SE(3)$, and nonlinear control systems are developed to track outputs of several flight modes [12]. Several aggressive maneuvers of a quadrotor UAV are also demonstrated based on a hybrid control architecture.

In this paper, we extend the results of [12] to construct nonlinear robust tracking control systems on $SE(3)$ for a quadrotor UAV. We assume that there exist unstructured, bounded uncertainties, with pre-determined bounds, on the translational dynamics and the rotation dynamics of a quadrotor UAV. Output tracking control systems are developed to follow an attitude command or a position command for the vehicle center of mass. We show that the tracking errors are uniformly ultimately bounded, and the size of the ultimate bound can be arbitrarily reduced. The robustness of the proposed tracking control systems are critical in generating complex maneuvers, as the impact of the several aerodynamic effects resulting from the variation in air speed is significant even at moderate velocities [13].

Compared with other tracking controllers considered in [14], [15], [16], the unique features of the control system proposed in this paper are as follows: (i) it is developed for the full six degrees of freedom dynamic model of a quadrotor UAV, (ii) a rigorous Lyapunov analysis, that explicitly considers the coupling between the translational dynamics and the rotational dynamics, is presented to establish stability properties, (iii) it is robust against unstructured uncertainties in both the translational dynamics and the rotational dynamics of a quadrotor UAV, and (iv) complicated reachability set analysis is not required to guarantee safe switching between different flight modes to generate complex maneuvers.

The paper is organized as follows. We develop a globally defined model for a quadrotor UAV in Section II. A hybrid control architecture is introduced and a robust attitude tracking control system is developed in Section III. Section IV present results for a robust position tracking, followed by numerical examples in Section V. Due to page limit, all of proofs are relegated to [17].

II. QUADROTOR DYNAMICS MODEL

Consider a quadrotor UAV model illustrated in Figure 1. This is a system of four identical rotors and propellers located at the vertices of a square, which generate a thrust and torque normal to the plane of this square. We choose an inertial reference frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and a body-fixed frame $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$. The origin of the body-fixed frame is located at the center of mass of this vehicle. The first and the second axes of the body-fixed frame, \vec{b}_1, \vec{b}_2 , lie in the plane defined by the centers of the four rotors, as illustrated

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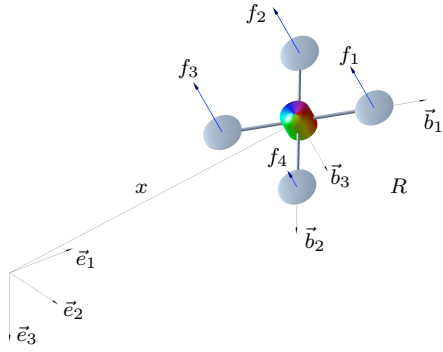


Fig. 1. Quadrotor model

in Figure 1. The third body-fixed axis \vec{b}_3 is normal to this plane. Each of the inertial reference frame and the body-fixed reference frame consist of a triad of orthogonal vectors defined according to the right hand rule. Define

| | |
|---------------------------------|--|
| $m \in \mathbb{R}$ | the total mass |
| $J \in \mathbb{R}^{3 \times 3}$ | the inertia matrix with respect to the body-fixed frame |
| $R \in \text{SO}(3)$ | the rotation matrix from the body-fixed frame to the inertial frame |
| $\Omega \in \mathbb{R}^3$ | the angular velocity in the body-fixed frame |
| $x \in \mathbb{R}^3$ | the position vector of the center of mass in the inertial frame |
| $v \in \mathbb{R}^3$ | the velocity vector of the center of mass in the inertial frame |
| $d \in \mathbb{R}$ | the distance from the center of mass to the center of each rotor in the \vec{b}_1, \vec{b}_2 plane |
| $f_i \in \mathbb{R}$ | the thrust generated by the i -th propeller along the $-\vec{b}_3$ axis |
| $\tau_i \in \mathbb{R}$ | the torque generated by the i -th propeller about the \vec{b}_3 axis |
| $f \in \mathbb{R}$ | the total thrust magnitude, i.e., $f = \sum_{i=1}^4 f_i$ |
| $M \in \mathbb{R}^3$ | the total moment vector in the body-fixed frame |

The configuration of this quadrotor UAV is defined by the location of the center of mass and the attitude with respect to the inertial frame. Therefore, the configuration manifold is the special Euclidean group $\text{SE}(3)$, which is the semidirect product of \mathbb{R}^3 and the special orthogonal group $\text{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} \mid R^T R = I, \det R = 1\}$.

The following conventions are assumed for the rotors and propellers, and the thrust and moment that they exert on the quadrotor UAV. We assume that the thrust of each propeller is directly controlled, and the direction of the thrust of each propeller is normal to the quadrotor plane. The first and third propellers are assumed to generate a thrust along the direction of $-\vec{b}_3$ when rotating clockwise; the second and fourth propellers are assumed to generate a thrust along the same direction of $-\vec{b}_3$ when rotating counterclockwise. Thus, the thrust magnitude is $f = \sum_{i=1}^4 f_i$, and it is

positive when the total thrust vector acts along $-\vec{b}_3$, and it is negative when the total thrust vector acts along \vec{b}_3 . By the definition of the rotation matrix $R \in \text{SO}(3)$, the total thrust vector is given by $-f R e_3 \in \mathbb{R}^3$ in the inertial frame. We also assume that the torque generated by each propeller is directly proportional to its thrust. Since it is assumed that the first and the third propellers rotate clockwise and the second and the fourth propellers rotate counterclockwise to generate a positive thrust along the direction of $-\vec{b}_3$, the torque generated by the i -th propeller about \vec{b}_3 can be written as $\tau_i = (-1)^i c_{\tau f} f_i$ for a fixed constant $c_{\tau f}$. All of these assumptions are common [18], [5].

Under these assumptions, the moment vector in the body-fixed frame is given by

$$\begin{bmatrix} f \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -d & 0 & d \\ d & 0 & -d & 0 \\ -c_{\tau f} & c_{\tau f} & -c_{\tau f} & c_{\tau f} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}. \quad (1)$$

The determinant of the above 4×4 matrix is $8c_{\tau f} d^2$, so it is invertible when $d \neq 0$ and $c_{\tau f} \neq 0$. Therefore, for given thrust magnitude f and given moment vector M , the thrust of each propeller f_1, f_2, f_3, f_4 can be obtained from (1). Using this equation, the thrust magnitude $f \in \mathbb{R}$ and the moment vector $M \in \mathbb{R}^3$ are viewed as control inputs in this paper.

The equations of motion can be written as

$$\dot{x} = v, \quad (2)$$

$$m\dot{v} = mge_3 - f R e_3 + \Delta_x, \quad (3)$$

$$\dot{R} = R \hat{\Omega}, \quad (4)$$

$$J \hat{\Omega} + \Omega \times J \Omega = M + \Delta_R, \quad (5)$$

where the *hat map* $\hat{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is defined by the condition that $\hat{x}y = x \times y$ for all $x, y \in \mathbb{R}^3$. This identifies the Lie algebra $\mathfrak{so}(3)$ with \mathbb{R}^3 using the vector cross product in \mathbb{R}^3 . The inverse of the hat map is denoted by the *vee map*, $\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$. Unstructured uncertainties in the translational dynamics and the rotational dynamics of a quadrotor UAV are denoted by Δ_x and $\Delta_R \in \mathbb{R}^3$, respectively. We assume that uncertainties are bounded:

$$\|\Delta_x\| \leq \delta_x, \quad \|\Delta_R\| \leq \delta_R \quad (6)$$

for known, positive constants $\delta_x, \delta_R \in \mathbb{R}$.

Throughout this paper, $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the minimum eigenvalue and the maximum eigenvalue of a matrix, respectively.

III. ATTITUDE CONTROLLED FLIGHT MODE

A. Flight Modes

Since the quadrotor UAV has four inputs, it is possible to achieve asymptotic output tracking for at most four quadrotor UAV outputs. The quadrotor UAV has three translational and three rotational degrees of freedom; it is not possible to achieve asymptotic output tracking of both attitude and position of the quadrotor UAV. This motivates us to introduce several flight modes, namely (1) an attitude controlled flight mode, and (2) a position controlled flight mode.

A complex flight maneuver can be defined by specifying a concatenation of flight modes together with conditions for switching between them; for each flight mode one also specifies the desired or commanded outputs as functions of time. Unlike a hybrid flight control system that requires reachability analyses [16], the proposed control system is robust to switching conditions since each flight mode has almost global stability properties, and it is straightforward to design a complex maneuver of a quadrotor UAV.

In this section, an attitude controlled flight mode is considered, where the outputs are the attitude of the quadrotor UAV and the controller for this flight mode achieves asymptotic attitude tracking.

B. Attitude Tracking Errors

Suppose that an arbitrary smooth attitude command $R_d(t) \in \text{SO}(3)$ is given. The corresponding angular velocity command is obtained by the attitude kinematics equation, $\hat{\Omega}_d = R_d^T \dot{R}_d$.

We first define errors associated with the attitude dynamics of the quadrotor UAV. The attitude error function studied in [9], [19], and several properties are summarized as follows.

Proposition 1: For a given tracking command (R_d, Ω_d) , and the current attitude and angular velocity (R, Ω) , we define an attitude error function $\Psi : \text{SO}(3) \times \text{SO}(3) \rightarrow \mathbb{R}$, an attitude error vector $e_R \in \mathbb{R}^3$, and an angular velocity error vector $e_\Omega \in \mathbb{R}^3$ as follows:

$$\Psi(R, R_d) = \frac{1}{2} \text{tr}[I - R_d^T R], \quad (7)$$

$$e_R = \frac{1}{2} (R_d^T R - R^T R_d)^\vee, \quad (8)$$

$$e_\Omega = \Omega - R^T R_d \Omega_d, \quad (9)$$

Then, the following statements hold:

- (i) Ψ is positive-definite about $R = R_d$.
- (ii) the left-trivialized derivative of Ψ is given by

$$\mathbf{T}_I^* \mathbf{L}_R(\mathbf{D}_R \Psi(R, R_d)) = e_R. \quad (10)$$

- (iii) the critical points of Ψ , where $e_R = 0$, are $\{R_d\} \cup \{R_d \exp(\pi \hat{s}), s \in \mathbb{S}^2\}$.
- (iv) a lower bound of Ψ is given as follows:

$$\frac{1}{2} \|e_R(R, R_d)\|^2 \leq \Psi(R, R_d), \quad (11)$$

- (v) Let ψ be a positive constant that is strictly less than 2. If $\Psi(R, R_d) < \psi < 2$, then an upper bound of Ψ is given by

$$\Psi(R, R_d) \leq \frac{1}{2 - \psi} \|e_R(R, R_d)\|^2. \quad (12)$$

Proof: See [19]. ■

C. Attitude Tracking Controller

We now introduce a nonlinear controller for the attitude controlled flight mode, described by an expression for the moment vector:

$$M = -k_R e_R - k_\Omega e_\Omega + \Omega \times J \Omega$$

$$-J(\hat{\Omega} R^T R_d \Omega_d - R^T R_d \hat{\Omega}_d) + \mu_R, \quad (13)$$

$$\mu_R = -\frac{\delta_R^2 e_A}{\delta_R \|e_A\| + \epsilon_R}, \quad (14)$$

$$e_A = e_\Omega + c_2 J^{-1} e_R, \quad (15)$$

where $k_R, k_\Omega, c_2, \epsilon_R$ are positive constants.

In this attitude controlled mode, it is possible to ignore the translational motion of the quadrotor UAV; consequently the reduced model for the attitude dynamics are given by equations (4), (5), using the controller expression (13)-(15). We now state the result that the tracking errors (e_R, e_Ω) are uniformly ultimately bounded.

Proposition 2: (Robustness of Attitude Controlled Flight Mode) Suppose that the initial attitude error satisfies

$$\Psi(R(0), R_d(0)) < \psi_2 < 2 \quad (16)$$

for a constant ψ_2 . Consider the control moment M defined in (13)-(15). For positive constants k_R, k_Ω , the constants c_2, ϵ_R are chosen such that

$$c_2 < \min \left\{ k_\Omega, \frac{4k_\Omega k_R \lambda_m(J)^2}{k_\Omega^2 \lambda_M(J) + 4k_R \lambda_m(J)^2}, \sqrt{k_R \lambda_m(J)} \right\}, \quad (17)$$

$$\epsilon_R < \frac{\lambda_m(M_{21}) \lambda_m(W_2)}{\lambda_M(M_{22})} \psi_2 (2 - \psi_2), \quad (18)$$

where the matrices $M_{21}, M_{22}, W_2 \in \mathbb{R}^{2 \times 2}$ are given by

$$M_{21} = \frac{1}{2} \begin{bmatrix} k_R & -c_2 \\ -c_2 & \lambda_m(J) \end{bmatrix}, \quad M_{22} = \frac{1}{2} \begin{bmatrix} \frac{2k_R}{2-\psi_2} & c_2 \\ c_2 & \lambda_M(J) \end{bmatrix},$$

$$W_2 = \begin{bmatrix} \frac{c_2 k_R}{\lambda_M(J)} & -\frac{c_2 k_\Omega}{2\lambda_m(J)} \\ -\frac{c_2 k_\Omega}{2\lambda_m(J)} & k_\Omega - c_2 \end{bmatrix}.$$

Then, the attitude tracking errors (e_R, e_Ω) are uniformly ultimately bounded, and the ultimate bound is given by

$$\left\{ \|e_R\|^2 + \|e_\Omega\|^2 \leq \frac{\lambda_M(M_{22})}{\lambda_m(M_{21}) \lambda_m(W_2)} \epsilon_R \right\}. \quad (19)$$

Proof: See [17]. ■

From (16), the initial attitude error should be less than 180° , in terms of the rotation angle about the eigenaxis between R and R_d . We can further show that the attitude tracking errors exponentially converges to (19), where the size of the ultimate bound can be reduced by the controller parameter ϵ_R . It is also possible to achieve exponential attractiveness if the constant ϵ_R in (14) is replaced by $\epsilon_R \exp(-\beta t)$ for $\beta > 0$. All of these results can be applied to a nonlinear robust control problem for the attitude dynamics of any rigid body.

Asymptotic tracking of the quadrotor attitude does not require specification of the thrust magnitude. As an auxiliary problem, the thrust magnitude can be chosen in many different ways to achieve an additional translational motion objective. For example, it can be used to asymptotically track a quadrotor altitude command [12].

Since the translational motion of the quadrotor UAV can only be partially controlled; this flight mode is most suitable for short time periods where an attitude maneuver is to be completed.

IV. POSITION CONTROLLED FLIGHT MODE

We now introduce a nonlinear controller for the position controlled flight mode. This flight mode requires analysis of the coupled translational and rotational equations of motion; hence, we make use of the notation and analysis in the prior section to describe the properties of the closed loop system in this flight mode.

A. Position Tracking Errors

An arbitrary smooth position tracking command $x_d(t) \in \mathbb{R}^3$ is chosen. The position tracking errors for the position and the velocity are given by:

$$e_x = x - x_d, \quad e_v = v - \dot{x}_d. \quad (20)$$

Following the prior definition of the attitude error and the angular velocity error, we define

$$e_R = \frac{1}{2}(R_c^T R - R^T R_c)^\vee, \quad e_\Omega = \Omega - R^T R_c \Omega_c, \quad (21)$$

and the computed attitude $R_c(t) \in \text{SO}(3)$ and computed angular velocity $\Omega_c \in \mathbb{R}^3$ are given by

$$R_c = [b_{1c}; b_{3c} \times b_{1c}; b_{3c}], \quad \hat{\Omega}_c = R_c^T \dot{R}_c, \quad (22)$$

where $b_{3c} \in \mathbb{S}^2$ is defined by

$$b_{3c} = -\frac{-k_x e_x - k_v e_v - m g e_3 + m \ddot{x}_d + \mu_x}{\| -k_x e_x - k_v e_v - m g e_3 + m \ddot{x}_d + \mu_x \|}, \quad (23)$$

and $b_{1c} \in \mathbb{S}^2$ is selected to be orthogonal to b_{3c} , thereby guaranteeing that $R_c \in \text{SO}(3)$. The constants k_x, k_v are positive, and the control input term μ_x is defined later in (28). We assume that

$$\| -k_x e_x - k_v e_v - m g e_3 + m \ddot{x}_d + \mu_x \| \neq 0, \quad (24)$$

and the commanded acceleration is uniformly bounded:

$$\| -m g e_3 + m \ddot{x}_d \| < B \quad (25)$$

for a given positive constant B .

B. Position Tracking Controller

The nonlinear controller for the position controlled flight mode, described by control expressions for the thrust magnitude and the moment vector, are:

$$f = (k_x e_x + k_v e_v + m g e_3 - m \ddot{x}_d - \mu_x) \cdot R e_3, \quad (26)$$

$$M = -k_R e_R - k_\Omega e_\Omega + \Omega \times J \Omega - J(\hat{\Omega} R^T R_c \Omega_c - R^T R_c \hat{\Omega}_c + \mu_R), \quad (27)$$

$$\mu_x = -\frac{\delta_x^{\tau+2} e_B \|e_B\|^\tau}{\delta_x^{\tau+1} \|e_B\|^{\tau+1} + \epsilon_x^{\tau+1}}, \quad (28)$$

$$e_B = e_v + \frac{c_1}{m} e_x, \quad (29)$$

$$\mu_R = -\frac{\delta_R^2 e_A}{\delta_R \|e_A\| + \epsilon_R}, \quad (30)$$

$$e_A = e_\Omega + c_2 J^{-1} e_R, \quad (31)$$

where $k_x, k_v, k_R, k_\Omega, c_1, c_2, \epsilon_x, \epsilon_R, \tau$ are positive constants, and $\tau > 2$.

The nonlinear controller given by equations (26), (27) can be given a backstepping interpretation. The computed attitude R_c given in equation (22) is selected so that the thrust axis $-b_3$ of the quadrotor UAV tracks the computed direction given by $-b_{3c}$ in (23), which is a direction of the thrust vector that achieves position tracking. The moment expression (27) causes the attitude of the quadrotor UAV to asymptotically track R_c and the thrust magnitude expression (26) achieves asymptotic position tracking.

The corresponding closed loop control system is described by equations (2)-(5), using the controller expressions (26)-(31). We now state the result that the tracking errors $(e_x, e_v, e_R, e_\Omega)$ are uniformly ultimately bounded.

Proposition 3: (Robustness of Position Controlled Flight Mode) Suppose that the initial conditions satisfy

$$\Psi(R(0), R_c(0)) < \psi_1 < 1, \quad (32)$$

$$\|e_x(0)\| < e_{x_{\max}}, \quad (33)$$

for positive constants $\psi_1, e_{x_{\max}}$. Consider the control inputs f, M defined in (26)-(31). For positive constants k_x, k_v , we choose positive constants $c_1, c_2, k_R, k_\Omega, \epsilon_x, \epsilon_R$ such that

$$c_1 < \min \left\{ k_v(1 - \alpha), \frac{4mk_x k_v(1 - \alpha)^2}{k_v^2(1 + \alpha)^2 + 4mk_x(1 - \alpha)}, \sqrt{k_x m} \right\}, \quad (34)$$

$$c_2 < \min \left\{ k_\Omega, \frac{4k_\Omega k_R \lambda_m(J)^2}{k_\Omega^2 \lambda_M(J) + 4k_R \lambda_m(J)^2}, \sqrt{k_R \lambda_m(J)} \right\}, \quad (35)$$

$$\lambda_m(W_2) > \frac{\|W_{12}\|^2}{4\lambda_m(W_1)}, \quad (36)$$

$$\epsilon_x + \epsilon_R <$$

$$\frac{\min\{\lambda_m(M_{11}), \lambda_m(M_{21})\} \min\{e_{x_{\max}}^2, \psi_1(2 - \psi_1)\}}{\max\{\lambda_M(M_{12}), \lambda_M(M'_{22})\}} \lambda_m(W), \quad (37)$$

where $\alpha = \sqrt{\psi_1(2 - \psi_1)}$, and the matrices $M_{11}, M_{12}, M_{21}, M'_{22}, W_1, W_{12}, W_2, W \in \mathbb{R}^{2 \times 2}$ are given by

$$M_{11} = \frac{1}{2} \begin{bmatrix} k_x & -c_1 \\ -c_1 & m \end{bmatrix}, \quad M_{12} = \frac{1}{2} \begin{bmatrix} k_x & c_1 \\ c_1 & m \end{bmatrix},$$

$$M_{21} = \frac{1}{2} \begin{bmatrix} k_R & -c_2 \\ -c_2 & \lambda_m(J) \end{bmatrix}, \quad M'_{22} = \frac{1}{2} \begin{bmatrix} \frac{2k_R}{2 - \psi_1} & c_2 \\ c_2 & \lambda_M(J) \end{bmatrix},$$

$$W_1 = \begin{bmatrix} \frac{c_1 k_x}{m}(1 - \alpha) & -\frac{c_1 k_v}{2m}(1 + \alpha) \\ -\frac{c_1 k_x}{2m}(1 + \alpha) & k_v(1 - \alpha) - c_1 \end{bmatrix},$$

$$W_{12} = \begin{bmatrix} \frac{c_1}{m}(B + \delta_x) & 0 \\ B + \delta_x + k_x e_{x_{\max}} & 0 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} \frac{c_2 k_R}{\lambda_m(J)} & -\frac{c_2 k_\Omega}{2\lambda_m(J)} \\ -\frac{c_2 k_\Omega}{2\lambda_m(J)} & k_\Omega - c_2 \end{bmatrix},$$

$$W = \begin{bmatrix} \lambda_m(W_1) & -\frac{1}{2} \|W_{12}\|_2 \\ -\frac{1}{2} \|W_{12}\|_2 & \lambda_m(W_2) \end{bmatrix}.$$

Then, the tracking errors $(e_x, e_v, e_R, e_\Omega)$ are uniformly ultimately bounded, and the ultimate bound is given by

$$\left\{ \|e_x\|^2 + \|e_v\|^2 + \|e_R\|^2 + \|e_\Omega\|^2 <$$

$$\frac{\max\{\lambda_M(M_{12}), \lambda_M(M'_{22})\}}{\min\{\lambda_m(M_{11}), \lambda_m(M_{21})\}\lambda_m(W)}(\epsilon_x + \epsilon_R)\}. \quad (38)$$

Proof: See [17]. ■

Proposition 3 requires that the initial attitude error is less than 90° in (32). Suppose that this is not satisfied, i.e. $1 \leq \Psi(R(0), R_c(0)) < 2$. We can still apply Proposition 2, which states that the attitude error exponentially decreases until it enters the ultimate bound given by (19). If the constant ϵ_R is sufficiently small, we can guarantee that the attitude error function decreases to satisfy (32) in a finite time. Therefore, by combining the results of Proposition 2 and 3, we can show ultimate boundedness of the tracking errors when $\Psi(R(0), R_c(0)) < 2$.

Proposition 4: (Robustness of Position Controlled Flight Mode with a Larger Initial Attitude Error) Suppose that the initial conditions satisfy

$$1 \leq \Psi(R(0), R_c(0)) < \psi_2 < 2, \quad (39)$$

$$\|e_x(0)\| < e_{x_{\max}}, \quad (40)$$

for a constant $\psi_2, e_{x_{\max}}$. Consider the control inputs f, M defined in (26)-(31), where the control parameters $k_x, k_v, k_R, k_\Omega, c_1, c_2, \epsilon_x, \epsilon_R$ satisfy (34)-(37) for a positive constant $\psi_1 < 1$. If the constant ϵ_R is sufficiently small such that

$$\epsilon_R < \frac{\lambda_m(M_{21})\lambda_m(W_2)}{\lambda_M(M_{22})}\psi_1(2 - \psi_1), \quad (41)$$

then the tracking errors $(e_x, e_v, e_R, e_\Omega)$ are uniformly ultimately bounded, and the ultimate bound is given by (38).

Proof: See [17]. ■

C. Direction of the First Body-Fixed Axis

As described above, the construction of the orthogonal matrix R_c involves having its third column b_{3_c} specified by a normalized feedback function, and its first column b_{1_c} is chosen to be orthogonal to the third column. The unit vector b_{1_c} can be arbitrarily chosen in the plane normal to b_{3_c} , which corresponds to a one-dimensional degree of choice.

By choosing b_{1_c} properly, we constrain the asymptotic direction of the first body-fixed axis. Here, we propose to specify the *projection* of the first body-fixed axis onto the plane normal to b_{3_c} . In particular, we choose a desired direction $b_{1_d} \in \mathbb{S}^2$, that is not parallel to b_{3_c} , and b_{1_c} is selected as $b_{1_c} = \text{Proj}[b_{1_d}]$, where $\text{Proj}[\cdot]$ denotes the normalized projection onto the plane perpendicular to b_{3_c} . In this case, the first body-fixed axis does not converge to b_{1_d} , but it converges to the projection of b_{1_d} , i.e. $b_1 \rightarrow b_{1_c} = \text{Proj}[b_{1_d}]$ as $t \rightarrow \infty$, up to the ultimate bound described by (38). In other words, the first body-fixed axis converges to a small neighborhood of the intersection of the plane normal to b_{3_c} and the plane spanned by b_{3_c} and b_{1_d} . This can be used to specify the heading direction of a quadrotor UAV in the horizontal plane (see Figure 2 and [12] for details).

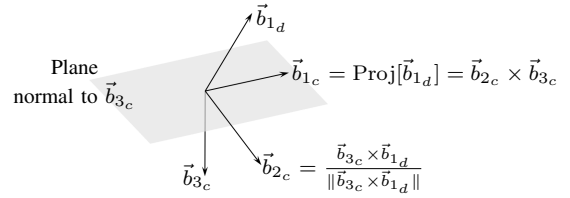


Fig. 2. Convergence property of the first body-fixed axis: in the proposed control system, b_{3_c} is determined by (23). We choose a desired direction of the first body fixed axis, namely b_{1_d} that is not parallel to b_{3_c} , and project it on to the plane normal to b_{3_c} to obtain b_{1_c} . This guarantees that the first body-fixed axis converges to b_{1_c} , and therefore it asymptotically lies in the plane spanned by b_{1_d} and b_{3_c} . As b_{3_c} converges to the direction of $ge_3 - \ddot{x}_d$ in (23), this allows us to specify the direction of the first body-fixed axis in the plane normal to $ge_3 - \ddot{x}_d$. For all cases, the ultimate convergence error is described by (38).

V. NUMERICAL EXAMPLES

Numerical results are presented to demonstrate the prior approach for performing complex flight maneuvers. The parameters are chosen to match a quadrotor UAV described in [20]: $J = [0.0820, 0.0845, 0.1377] \text{ kg} - \text{m}^2$, $m = 4.34 \text{ kg}$, $d = 0.315 \text{ m}$, $c_{\tau f} = 8.004 \times 10^{-3} \text{ m}$.

The controller parameters are chosen as follows: $k_x = 59.02$, $k_v = 24.30$, $k_R = 8.81$, $k_\Omega = 1.54$, $c_1 = 3.6$, $c_2 = 0.6$, $\epsilon_x = \epsilon_R = 0.04$. We consider a fixed disturbance for the translational dynamics, and an oscillatory disturbance for the rotational dynamics as follows:

$$\Delta_x = [2.50, 1.25, 2.00]^T \text{ N},$$

$$\Delta_R(t) = \frac{2}{\sqrt{3}}[\sin(8\pi t), \sin(\pi t), \cos(4\pi t)]^T \text{ Nm}.$$

The corresponding bounds of the disturbances are given by $\delta_x = 4.34$ and $\delta_R = 2$. We consider the following two cases.

Case I (elliptic helix): The initial conditions are

$$x(0) = [0.1, 0, 0]^T \text{ m}, \quad v(0) = [0, 0, 0]^T \text{ m/s},$$

$$R(0) = I, \quad \Omega(0) = [0, 0, 0]^T \text{ rad/s}.$$

The desired position command is an elliptic helix, given by

$$x_d(t) = [0.4t, 0.4 \sin(\pi t), -0.6 \cos(\pi t)]^T \text{ m},$$

and the desired heading direction is fixed as $\vec{b}_{1_d} = [1, 0, 0]^T$. This corresponds to the position controlled flight mode described in Proposition 3, as the initial attitude error is $\Psi(0) = 0.14 < 1$.

Figure 3 shows simulation results, where the position tracking error converges to a small neighborhood of the zero tracking errors, and the terminal tracking error is 1.2 cm. In [17], numerical results with no robust control input terms are given as a comparison, where it is observed that the angular velocity tracking error is mostly driven by the disturbance Δ_R , and the corresponding position tracking error is larger than 0.1 m. This illustrates the robustness of the proposed control system for a complex maneuver with larger disturbances.

Case II (hovering): The initial conditions are given by

$$x(0) = [0.1, 0, 0]^T \text{ m}, \quad v(0) = [0, 0, 0]^T \text{ m/s},$$

$$R(0) = \exp(0.99\pi \hat{e}_1), \quad \Omega(0) = [0, 0, 0]^T \text{ rad/s},$$

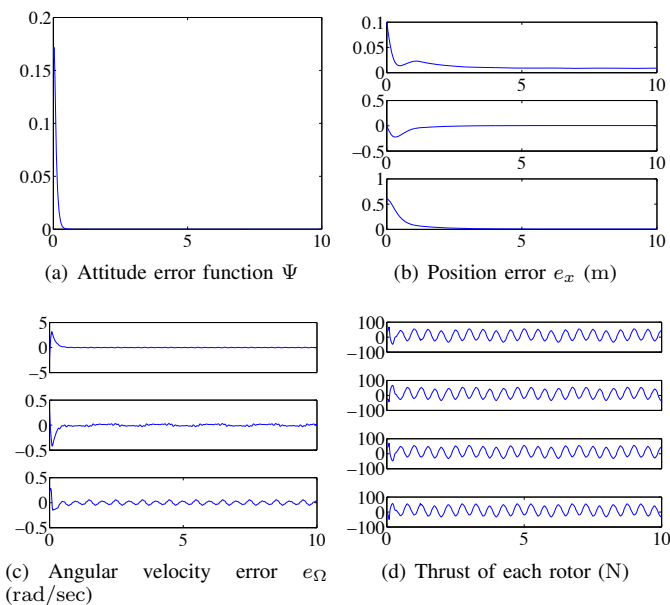


Fig. 3. Case I: robust position controlled flight mode to follow an elliptic helix

where $e_1 = [1, 0, 0] \in \mathbb{R}^3$. The desired position command is

$$x_d(t) = [0, 0, 0]^T \text{ m,}$$

and the desired heading direction is fixed as $\vec{b}_{1_d} = [1, 0, 0]^T$. This describes a case that a quadrotor UAV should recover from an initially upside-down configuration.

The initial attitude error is given by $1 \leq (\Psi(0) = 1.9995) < 2$, and therefore, it corresponds to Proposition 4 that is based on both of the attitude controlled flight mode and the position controlled flight mode. Figure 4 illustrates excellent convergence properties of the proposed control system for a large initial attitude error, where the terminal position tracking error is 1.2 cm.

REFERENCES

- [1] S. Bouabdalla and R. Siegward, "Backstepping and sliding-mode techniques applied to an indoor micro quadrotor," in *Proceedings of the IEEE International Conference on Robotics and Automation*, 2005, pp. 2259–2264.
- [2] M. Efe, "Robust low altitude behavior control of a quadrotor rotorcraft through sliding modes," in *Proceedings of the Mediterranean Conference on Control and Automation*, 2007.
- [3] G. Raffo, M. Ortega, and F. Rubio, "An integral predictive/nonlinear H_∞ control structure for a quadrotor helicopter," *Automatica*, vol. 46, pp. 29–30, 2010.
- [4] C. Nicol, C. Macnab, and A. Ramirez-Serrano, "Robust neural network control of a quadrotor helicopter," in *Proceedings of the Canadian Conference on Electrical and Computer Engineering*, 2008, pp. 1233–1237.
- [5] A. Tayebi and S. McGilvray, "Attitude stabilization of a VTOL quadrotor aircraft," *IEEE Transactions on Control System Technology*, vol. 14, no. 3, pp. 562–571, 2006.
- [6] C. Mayhew, R. Sanfelice, and A. Teel, "Quaternion-based hybrid control for robust global attitude tracking," *IEEE Transactions on Automatic Control*, 2011.
- [7] —, "On the non-robustness of inconsistent quaternion-based attitude control systems using memoryless path-lifting schemes," in *Proceeding of the American Control Conference*, 2011.

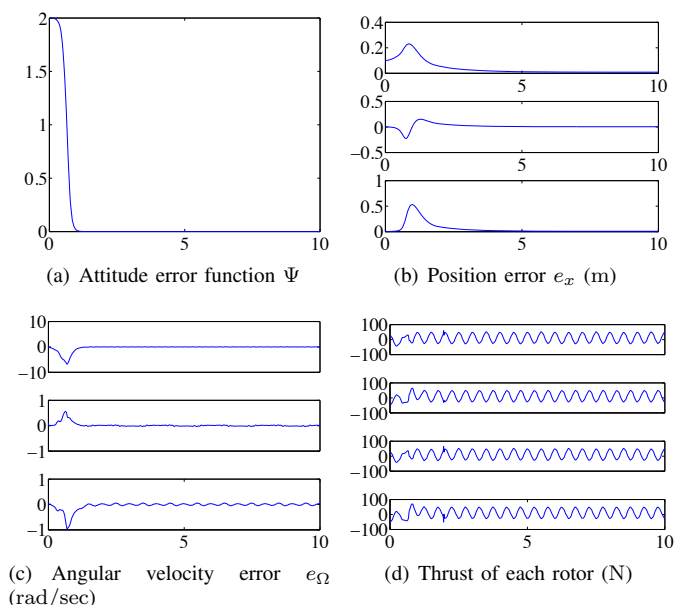


Fig. 4. Case II: robust position controlled flight mode to recover from an initially upside-down configuration

- [8] S. Bhat and D. Bernstein, "A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon," *Systems and Control Letters*, vol. 39, no. 1, pp. 66–73, 2000.
- [9] F. Bullo and A. Lewis, *Geometric control of mechanical systems*, ser. Texts in Applied Mathematics. New York: Springer-Verlag, 2005, vol. 49, modeling, analysis, and design for simple mechanical control systems.
- [10] D. Cabecinhas, R. Cunha, and C. Silvestre, "Output-feedback control for almost global stabilization of fully-actuated rigid bodies," in *Proceedings of IEEE Conference on Decision and Control*, 3583–3588, Ed., 2008.
- [11] N. Chaturvedi, A. Sanyal, and N. McClamroch, "Rigid-body attitude control," *IEEE Control Systems Magazine*, vol. 31, no. 3, pp. 30–51, 2011.
- [12] T. Lee, M. Leok, and N. McClamroch, "Control of complex maneuvers for a quadrotor UAV using geometric methods on $SE(3)$," arXiv. [Online]. Available: <http://arxiv.org/abs/1003.2005>
- [13] G. Hoffmann, H. Huang, S. Waslander, and C. Tomlin, "Quadrotor helicopter flight dynamics and control: Theory and experiment," in *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, 2007, AIAA 2007-6461.
- [14] D. Cabecinhas, R. Cunha, and C. Silvestre, "Rotorcraft path following control for extended flight envelope coverage," in *Proceedings of the IEEE Conference on Decision and Control*, 2009, pp. 3460–3465.
- [15] M. Hua, T. Hamel, P. Morin, and C. Samson, "A control approach for thrust-propelled underactuated vehicles and its application to VTOL drones," *IEEE Transactions on Automatic Control*, vol. 54, no. 8, pp. 1834–1853, 2009.
- [16] J. Gillula, G. Hoffmann, H. Huang, M. Vitus, and C. Tomlin, "Applications of hybrid reachability analysis to robotic aerial vehicles," *The International Journal of Robotics Research*, vol. 30, no. 3, pp. 335–354, 2011.
- [17] T. Lee, M. Leok, and N. McClamroch, "Nonlinear robust tracking control of a quadrotor UAV on $SE(3)$," arXiv. [Online]. Available: <http://arxiv.org/abs/1109.4457>
- [18] P. Castillo, R. Lozano, and A. Dzul, "Stabilization of a mini rotorcraft with four rotors," *IEEE Control System Magazine*, pp. 45–55, 2005.
- [19] T. Lee, "Robust adaptive geometric tracking controls on $SO(3)$ with an application to the attitude dynamics of a quadrotor UAV," arXiv, 2011. [Online]. Available: <http://arxiv.org/abs/1108.6031>
- [20] P. Pounds, R. Mahony, and P. Corke, "Modeling and control of a large quadrotor robot," *Control Engineering Practice*, vol. 18, pp. 691–699, 2010.