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Lie group variational integrators for the full body problem

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Abstract

We develop the equations of motion for full body models that describe the dynamics of rigid bodies, acting under their mutual gravity. The equations are derived using a variational approach where variations are defined on the Lie group of rigid body configurations. Both continuous equations of motion and variational integrators are developed in Lagrangian and Hamiltonian forms, and the reduction from the inertial frame to a relative frame is also carried out. The Lie group variational integrators are shown to be symplectic, to preserve conserved quantities, and to guarantee exact evolution on the configuration space. One of these variational integrators is used to simulate the dynamics of two rigid dumbbell bodies.
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1. Introduction

1.1. Overview

The full body problem studies the dynamics of rigid bodies interacting under their mutual potential, and the mutual potential of distributed rigid bodies depends on both the position and the attitude of the bodies. Therefore, the translational and the rotational dynamics are coupled in the full body problem. The full body problem arises in numerous engineering and scientific fields. For example, in astrodynamics, the trajectory of a large spacecraft around the Earth is affected by the attitude of the spacecraft, and the dynamics of a binary asteroid pair is characterized by the non-spherical mass distributions of the

bodies. In chemistry, the full rigid body model is used to study molecular dynamics. The importance of the full body problem is summarized in [1], along with a preliminary discussion from the point of view of geometric mechanics.

The full two body problem was studied by Maciejewski [2], and he presented equations of motion in inertial and relative frames and discussed the existence of relative equilibria in the system. However, he does not derive the equations of motion, nor does he discuss the reconstruction equations that allow the recovery of the inertial dynamics from the relative dynamics. Scheeres derived a stability condition for the full two body problem [3], and he studied the planar stability of an ellipsoid-sphere model [4]. Recently, interest in the full body problem has increased, as it is estimated that up to 16% of near-earth asteroids are binaries [5]. Spacecraft motion about binary asteroids have been discussed using the restricted three body model [6,7], and the four body model [8].

Conservation laws are important for studying the dynamics of the full body problem, because they describe qualitative characteristics of the system dynamics. The representation used for the attitude of the bodies should be globally defined since the complicated dynamics of such systems would require frequent coordinate changes when

using representations that are only defined locally. General numerical integration methods, such as the Runge–Kutta scheme, do not preserve first integrals nor the exact geometry of the full body dynamics [9]. A more careful analysis of computational methods used to study the full body problem is crucial.

Variational integrators and Lie group methods provide a systematic method of constructing structure-preserving numerical integrators [9]. The idea of the variational approach is to discretize Hamilton's principle rather than the continuous equations of motion [10]. The numerical integrator obtained from the discrete Hamilton's principle exhibits excellent energy properties [11], conserves first integrals associated with symmetries by a discrete version of Noether's theorem, and preserves the symplectic structure. Many interesting differential equations, including full body dynamics, evolve on a Lie group. Lie group methods consist of numerical integrators that preserve the geometry of the configuration space by automatically remaining on the Lie group [12].

Moser and Vesselov [13], Wendlandt and Marsden [14] developed numerical integrators for a free rigid body by imposing an orthogonal constraint on the attitude variables and by using unit quaternions, respectively. The idea of using the Lie group structure and the exponential map to numerically compute rigid body dynamics arises in the work of Simo et al. [15], and in the work by Krysl [16]. A Lie group approach is explicitly adopted by Lee, Leok, and McClamroch in the context of a variational integrator for rigid body attitude dynamics with a potential dependent on the attitude, namely the 3D pendulum dynamics, in [17].

The motion of full rigid bodies depends essentially on the mutual gravitational potential, which in turn depends only on the relative positions and the relative attitudes of the bodies. Marsden et al. introduce discrete Euler–Poincaré and Lie–Poisson equations in [18,19]. They reduce the discrete dynamics on a Lie group to the dynamics on the corresponding Lie algebra. Sanyal, Shen and McClamroch develop variational integrators for mechanical systems with configuration dependent inertia and they perform discrete Routh reduction in [20]. A more intrinsic development of discrete Routh reduction is given in [21,22].

1.2. Contributions

The purpose of this paper is to provide a complete set of equations of motion for the full body problem. The equations of motion are categorized by three independent properties: continuous/discrete time, inertial/relative frame, and Lagrangian/Hamiltonian forms. Therefore, a total of eight types of equations of motion for the full body problem are given in this paper. The relationships between these equations of motion are shown in Fig. 1, and are further summarized in Fig. 7.

Continuous equations of motion for the full body problem are given in [2] without any formal derivation of the
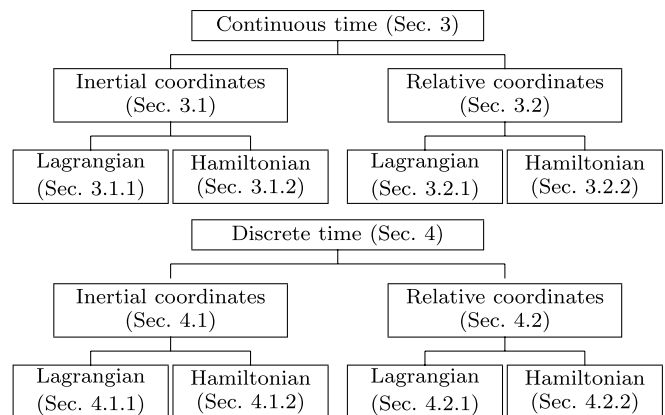

Fig. 1. Eight types of equations of motion for the full body problem.

equations. We show, in this paper, that the equations of motion for the full body problem can be derived from Hamilton's principle. The proper form for the variations of Lie group elements in the configuration space lead to a systematic derivation of the equations of motion.

This paper develops discrete variational equations of motion for the full body model following a similar variational approach but carried out within a discrete time framework. Since numerical integrators are derived from the discrete Hamilton's principle, they exhibit symplectic and momentum preserving properties, and good energy behavior. They are also constructed to conserve the Lie group geometry on the configuration space. Numerical simulation results for the interaction of two rigid dumbbell models are given.

This paper is organized as follows. The basic idea of the variational integrator is given in Section 2. The continuous equations of motion and variational integrators are developed in Section 3 and 4. Numerical simulation results are given in Section 5. An appendix contains several technical details that are utilized in the main development.

2. Background

2.1. Hamilton's principle and variational integrators

The procedure to derive the Euler–Lagrange equations and Hamilton's equations from Hamilton's principle is shown in Fig. 2. When deriving the equations of motion, we first choose generalized coordinates q, and a corresponding configuration space Q. We then construct a Lagrangian from the kinetic and the potential energy. An action integral $\mathfrak{G} = \int_{t_0}^{t_f} L(q, \dot{q}) \, dt$ is defined as the path integral of the Lagrangian along a time-parameterized trajectory. After taking the variation of the action integral, and requiring it to be stationary, we obtain the Euler–Lagrange equations. If we use the Legendre transformation defined as

$$p \cdot \delta \dot{q} = \mathbb{F}L(q, \dot{q}) \cdot \delta \dot{q} = \left. \frac{\mathrm{d}}{\mathrm{d}\epsilon} \right|_{\epsilon=0} L(q, \dot{q} + \epsilon \delta \dot{q}), \qquad (1)$$
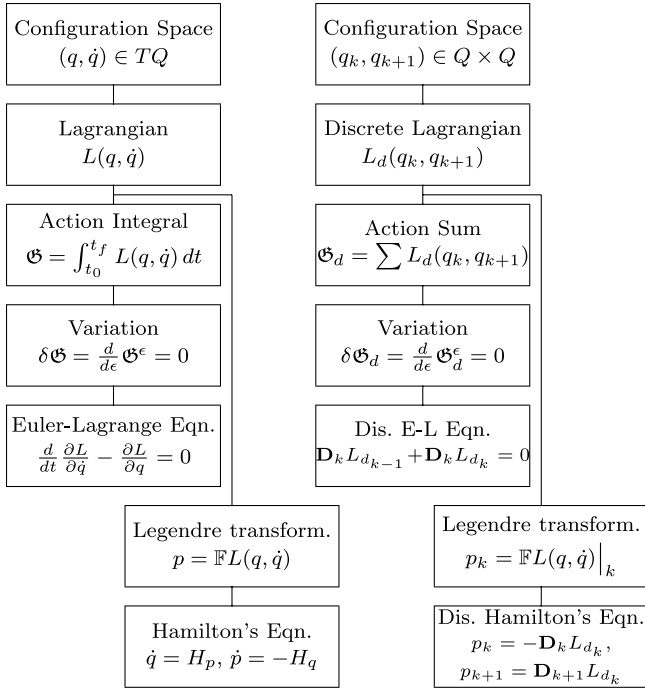
Fig. 2. Procedures to derive the continuous and discrete equations of motion.

where $\delta \dot{q} \in T_q Q$, then we obtain Hamilton's equations in terms of momenta variables $p = \mathbb{F}L(q, \dot{q}) \in T^*Q$. These equations are equivalent to the Euler–Lagrange equations [23].

There are two issues that arise in applying this procedure to the full body problem. The first is that the configuration space for each rigid body is the semi-direct product, $SE(3) = \mathbb{R}^3 \, \textcircled{s} \, SO(3)$, where $SO(3)$ is the Lie group of orthogonal matrices with determinant 1. Therefore, variations should be carefully chosen such that they respect the geometry of the configuration space. For example, a varied rotation matrix $R^\epsilon \in SO(3)$ can be expressed as

$$R^\epsilon = Re^{\epsilon\eta},$$

where $\epsilon \in \mathbb{R}$, and $\eta \in \mathfrak{so}(3)$ is a variation represented by a skew-symmetric matrix. The variation of the rotation matrix δR is the part of R^ϵ that is linear in ϵ

$$R^\epsilon = R + \epsilon \delta R + \mathcal{O}(\epsilon^2).$$

More precisely, δR is given by

$$\delta R = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} R^\epsilon = R\eta. \tag{2}$$

The second issue is that reduced variables can be used to obtain equations of motion expressed in relative coordinates. The variations of the reduced variables are constrained as they must arise from the variations of the unreduced variables while respecting the geometry of the configuration space. The proper variations of Lie group elements and reduced quantities are computed while deriving the continuous equations of motion.

Generally, numerical integrators are obtained by approximating the continuous Euler–Lagrange equation using a finite difference rule such as $\dot{q}_k = (q_{k+1} - q_k)/h$, where q_k denotes the value of $q(t)$ at $t = hk$ for an integration step size $h \in \mathbb{R}$ and an integer k. A variational integrator is derived by following a procedure shown in the right column of Fig. 2. When deriving a variational integrator, the velocity phase space $(q, \dot{q}) \in TQ$ of the continuous Lagrangian is replaced by $(q_k, q_{k+1}) \in Q \times Q$, and the discrete Lagrangian L_d is chosen such that it approximates a segment of the action integral

$$L_d(q_k, q_{k+1}) \approx \int_0^h L(q_{k,k+1}(t), \dot{q}_{k,k+1}(t)) \, \mathrm{d}t,$$

where $q_{k,k+1}(t)$ is the solution of the Euler–Lagrange equation satisfying boundary conditions $q_{k,k+1}(0) = q_k$ and $q_{k,k+1}(h) = q_{k+1}$. Then, the discrete action sum $\mathfrak{G}_d = \sum L_d(q_k, q_{k+1})$ approximates the action integral $\mathfrak{G}$. Taking the variations of the action sum, we obtain the discrete Euler–Lagrange equation

$$\mathbf{D}_{q_k}L_d(q_{k-1}, q_k) + \mathbf{D}_{q_k}L_d(q_k, q_{k+1}) = 0,$$

where $\mathbf{D}_{q_k}L_d$ denotes the partial derivative of L_d with respect to q_k. This yields a discrete Lagrangian map $F_{L_d}: (q_{k-1}, q_k) \mapsto (q_k, q_{k+1})$. Using a discrete analogue of the Legendre transformation, referred to as a discrete fiber derivative $\mathbb{F}L_d: Q \times Q \to T^*Q$, variational integrators can be expressed in Hamiltonian form as

$$p_k = -\mathbf{D}_{q_k}L_d(q_k, q_{k+1}), \tag{3}$$

$$p_{k+1} = \mathbf{D}_{q_{k+1}}L_d(q_k, q_{k+1}). \tag{4}$$

This yields a discrete Hamiltonian map $\tilde{F}_{L_d}: (q_k, p_k) \mapsto (q_{k+1}, p_{k+1})$. A complete development of variational integrators can be found in [10].

2.2. Notation

Variables in the inertial and the body fixed frames are denoted by lower-case and capital letters, respectively. A subscript i is used for variables related to the ith body. The relative variables have no subscript and the kth discrete variables have the second level subscript k. The letters x, v, Ω and R are used to denote the position, velocity, angular velocity and rotation matrix, respectively.

The trace of $A \in \mathbb{R}^{n \times n}$ is denoted by

$$\mathrm{tr}[A] = \sum_{i=1}^n [A]_{ii},$$

where $[A]_{ii}$ is the i, ith element of A. It can be shown that

$$\mathrm{tr}[AB] = \mathrm{tr}[BA] = \mathrm{tr}[B^{\mathrm{T}}A^{\mathrm{T}}] = \mathrm{tr}[A^{\mathrm{T}}B^{\mathrm{T}}], \tag{5}$$

$$\mathrm{tr}[A^{\mathrm{T}}B] = \sum_{p,q=1}^3 [A]_{pq}[B]_{pq}, \tag{6}$$

$$\mathrm{tr}[PQ] = 0 \tag{7}$$

for matrices $A, B \in \mathbb{R}^{n \times n}$, a skew-symmetric matrix $P = -P^T \in \mathbb{R}^{n \times n}$, and a symmetric matrix $Q = Q^T \in \mathbb{R}^{n \times n}$.

The map $S : \mathbb{R}^3 \mapsto \mathbb{R}^{3 \times 3}$ is defined by the condition that $S(x)y = x \times y$ for $x, y \in \mathbb{R}^3$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $S(x)$ is given by

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

It can be shown that

$$S(x)^T = -S(x), \tag{8}$$

$$S(x \times y) = S(x)S(y) - S(y)S(x) = yx^T - xy^T, \tag{9}$$

$$S(Rx) = RS(x)R^T, \tag{10}$$

$$S(x)^T S(x) = (x^T x)I_{3 \times 3} - xx^T = \text{tr}[xx^T]I_{3 \times 3} - xx^T \tag{11}$$

for $x, y \in \mathbb{R}^3$ and $R \in \text{SO}(3)$.

Using homogeneous coordinates, we can represent an element of SE(3) as follows:

$$\begin{bmatrix} R & x \\ 0 & 1 \end{bmatrix} \in \text{SE}(3)$$

for $x \in \mathbb{R}^3$ and $R \in \text{SO}(3)$. Then, an action on SE(3) is given by the usual matrix multiplication in $\mathbb{R}^{4 \times 4}$. For example

$$\begin{bmatrix} R_1 & x_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & x_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 x_2 + x_1 \\ 0 & 1 \end{bmatrix}$$

for $x_1, x_2 \in \mathbb{R}^3$ and $R_1, R_2 \in \text{SO}(3)$.

The action of an element of SE(3) on $\mathbb{R}^3$ can be expressed using a matrix–vector product, once we identify $\mathbb{R}^3$ with $\mathbb{R}^3 \times \{1\} \subset \mathbb{R}^4$. In particular, we see from

$$\begin{bmatrix} R & x_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} Rx_2 + x_1 \\ 1 \end{bmatrix}$$

that $x_2 \mapsto Rx_2 + x_1$.

3. Continuous time full body models

In this section, the continuous time equations of motion for the full body problem are derived in inertial and relative frames, and are expressed in both Lagrangian and in Hamiltonian form.

We define $O - e_1 e_2 e_3$ as an inertial frame, and $O_{\mathscr{B}_i} - E_{i_1} E_{i_2} E_{i_3}$ as a body fixed frame for the ith body, $\mathscr{B}_i$. The origin of the ith body fixed frame is located at the center of mass of body $\mathscr{B}_i$. The configuration space of the ith rigid body is $\text{SE}(3) = \mathbb{R}^3 \circledS \text{SO}(3)$. We denote the position of the center of mass of $\mathscr{B}_i$ in the inertial frame by $x_i \in \mathbb{R}^3$, and we denote the attitude of $\mathscr{B}_i$ by $R_i \in \text{SO}(3)$, which is a rotation matrix from the ith body fixed frame to the inertial frame.

3.1. Inertial frame

Lagrangian: To derive the equations of motion, we first construct a Lagrangian for the full body problem. Given

$(x_i, R_i) \in \text{SE}(3)$, the inertial position of a mass element of $\mathscr{B}_i$ is given by $x_i + R_i \rho_i$, where $\rho_i \in \mathbb{R}^3$ denotes the position of the mass element in the body fixed frame. Then, the kinetic energy of $\mathscr{B}_i$ can be written as

$$T_i = \frac{1}{2} \int_{\mathscr{B}_i} \| \dot{x}_i + \dot{R}_i \rho_i \|^2 \mathrm{d}m_i.$$

Using the fact that $\int_{\mathscr{B}_i} \rho_i \mathrm{d}m_i = 0$ and the kinematic equation $\dot{R}_i = R_i S(\Omega_i)$, the kinetic energy T_i can be rewritten as

$$T_i(\dot{x}_i, \Omega_i) = \frac{1}{2} \int_{\mathscr{B}_i} \| \dot{x}_i \|^2 + \| S(\Omega_i)\rho_i \|^2 \mathrm{d}m_i$$

$$= \frac{1}{2} m_i \| \dot{x}_i \|^2 + \frac{1}{2} \text{tr}[S(\Omega_i)J_{d_i}S(\Omega_i)^T], \tag{12}$$

where $m_i \in \mathbb{R}$ is the total mass of $\mathscr{B}_i$, $\Omega_i \in \mathbb{R}^3$ is the angular velocity of $\mathscr{B}_i$ in the body fixed frame, and $J_{d_i} = \int_{\mathscr{B}_i} \rho_i \rho_i^T \mathrm{d}m_i \in \mathbb{R}^{3 \times 3}$ is a nonstandard moment of inertia matrix. Using (11), it can be shown that the standard moment of inertia matrix $J_i = \int_{\mathscr{B}_i} S(\rho_i)^T S(\rho_i) \mathrm{d}m_i \in \mathbb{R}^{3 \times 3}$ is related to J_{d_i} by

$$J_i = \text{tr}[J_{d_i}]I_{3 \times 3} - J_{d_i},$$

and that the following condition holds for any $\Omega_i \in \mathbb{R}^3$

$$S(J_i \Omega_i) = S(\Omega_i)J_{d_i} + J_{d_i}S(\Omega_i). \tag{13}$$

We first derive equations using the nonstandard moment of inertia matrix J_{d_i}, and then express them in terms of the standard moment of inertia J_i by using (13).

The gravitational potential energy $U : \text{SE}(3)^n \mapsto \mathbb{R}$ is given by

$$U(x_1, x_2, \ldots, x_n, R_1, R_2, \ldots, R_n)$$
$$= -\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \int_{\mathscr{B}_i} \int_{\mathscr{B}_j} \frac{G\mathrm{d}m_i \mathrm{d}m_j}{\| x_i + R_i \rho_i - x_j - R_j \rho_j \|}, \tag{14}$$

where G is the universal gravitational constant.

Then, the Lagrangian for n full bodies can be written as

$$L(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{R}, \mathbf{\Omega}) = \sum_{i=1}^{n} \frac{1}{2} m_i \| \dot{x}_i \|^2 + \frac{1}{2} \text{tr}[S(\Omega_i)J_{d_i}S(\Omega_i)^T] - U(\mathbf{x}, \mathbf{R}), \tag{15}$$

where bold type letters denote ordered n-tuples of variables. For example, $\mathbf{x} \in (\mathbb{R}^3)^n$, $\mathbf{R} \in \text{SO}(3)^n$, and $\mathbf{\Omega} \in (\mathbb{R}^3)^n$ are defined as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{R} = (R_1, R_2, \ldots, R_n)$, and $\mathbf{\Omega} = (\Omega_1, \Omega_2, \ldots, \Omega_n)$, respectively.

Variations of variables: Since the configuration space is $\text{SE}(3)^n$, the variations should be carefully chosen such that they respect the geometry of the configuration space. The variations of $x_i : [t_0, t_f] \mapsto \mathbb{R}^3$ and $\dot{x}_i : [t_0, t_f] \mapsto \mathbb{R}^3$ are trivial, namely

$$x_i^\epsilon = x_i + \epsilon \delta x_i + \mathcal{O}(\epsilon^2),$$
$$\dot{x}_i^\epsilon = \dot{x}_i + \epsilon \delta \dot{x}_i + \mathcal{O}(\epsilon^2),$$

where $\delta x_i : [t_0, t_f] \mapsto \mathbb{R}^3$, $\delta \dot{x}_i : [t_0, t_f] \mapsto \mathbb{R}^3$ vanish at the initial time t_0 and at the final time t_f. The variation of $R_i : [t_0, t_f] \mapsto SO(3)$, as given in (2), is

$$\delta R_i = R_i \eta_i, \tag{16}$$

where $\eta_i : [t_0, t_f] \mapsto \mathfrak{so}(3)$ is a variation with values represented by a skew-symmetric matrix ($\eta_i^T = -\eta_i$) vanishing at t_0 and t_f. The variation of Ω_i can be computed from the kinematic equation $\dot{R}_i = R_i S(\Omega_i)$ and (16) to be

$$S(\delta \Omega_i) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} R_i^{\epsilon T} \dot{R}_i^{\epsilon} = \delta R_i^T \dot{R}_i + R_i^T \delta \dot{R}_i$$
$$= -\eta_i S(\Omega_i) + S(\Omega_i)\eta_i + \dot{\eta}_i. \tag{17}$$

3.1.1. Equations of motion: Lagrangian form

If we take variations of the Lagrangian using (16) and (17), we obtain the equations of motion from Hamilton's principle. We first take the variation of the kinetic energy of $\mathscr{B}_i$

$$\delta T_i = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} T_i(\dot{x}_i + \epsilon \delta \dot{x}_i, \Omega_i + \epsilon \delta \Omega_i)$$
$$= m_i \dot{x}_i^T \delta \dot{x}_i + \frac{1}{2} \mathrm{tr}[S(\delta \Omega_i) J_{d_i} S(\Omega_i)^T + S(\Omega_i) J_{d_i} S(\delta \Omega_i)^T].$$

Substituting (17) into the above equation and using (5), we obtain

$$\delta T_i = m_i \dot{x}_i^T \delta \dot{x}_i + \frac{1}{2} \mathrm{tr}[-\dot{\eta}_i \{ J_{d_i} S(\Omega_i) + S(\Omega_i) J_{d_i} \}]$$
$$+ \frac{1}{2} \mathrm{tr}[\eta_i S(\Omega_i) \{ J_{d_i} S(\Omega_i) + S(\Omega_i) J_{d_i} \}]$$
$$- \frac{1}{2} \mathrm{tr}[\eta_i \{ J_{d_i} S(\Omega_i) + S(\Omega_i) J_{d_i} \} S(\Omega_i)].$$

Using (9) and (13), δT_i is given by

$$\delta T_i = m_i \dot{x}_i^T \delta \dot{x}_i + \frac{1}{2} \mathrm{tr}[-\dot{\eta}_i S(J_i \Omega_i) + \eta_i S(\Omega_i \times J_i \Omega_i)]. \tag{18}$$

The variation of the potential energy is given by

$$\delta U = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} U(\mathbf{x} + \epsilon \delta \mathbf{x}, \mathbf{R} + \epsilon \delta \mathbf{R}),$$

where $\delta \mathbf{x} = (\delta x_1, \delta x_2, \ldots, \delta x_n)$, $\delta \mathbf{R} = (\delta R_1, \delta R_2, \ldots, \delta R_n)$. Then, δU can be written as

$$\delta U = \sum_{i=1}^{n} \left(\sum_{p=1}^{3} \frac{\partial U}{\partial [x_i]_p} [\delta x_i]_p + \sum_{p,q=1}^{3} \frac{\partial U}{\partial [R_i]_{pq}} [R_i \eta_i]_{pq} \right)$$
$$= \sum_{i=1}^{n} \left(\frac{\partial U^T}{\partial x_i} \delta x_i - \mathrm{tr} \left[\eta_i R_i^T \frac{\partial U}{\partial R_i} \right] \right), \tag{19}$$

where $[A]_{pq}$ denotes the p, qth element of a matrix A, and $\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial R_i}$ are given by $\left[\frac{\partial U}{\partial x_i} \right]_p = \frac{\partial U}{\partial [x_i]_p}$, $\left[\frac{\partial U}{\partial R_i} \right]_{pq} = \frac{\partial U}{\partial [R_i]_{pq}}$, respectively. The variation of the Lagrangian has the form

$$\delta L = \sum_{i=1}^{n} \delta T_i - \delta U, \tag{20}$$

which can be written more explicitly by using (18) and (19).

The action integral is defined to be

$$\mathfrak{G} = \int_{t_0}^{t_f} L(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{R}, \mathbf{\Omega}) \, dt. \tag{21}$$

The variation of the action integral can be written as

$$\delta \mathfrak{G} = \sum_{i=1}^{n} \int_{t_0}^{t_f} m_i \dot{x}_i^T \delta \dot{x}_i - \frac{\partial U^T}{\partial x_i} \delta x_i + \frac{1}{2} \mathrm{tr}[-\dot{\eta}_i S(J_i \Omega_i)]$$
$$+ \frac{1}{2} \mathrm{tr} \left[\eta_i \left\{ S(\Omega_i \times J_i \Omega_i) + 2 R_i^T \frac{\partial U}{\partial R_i} \right\} \right] dt.$$

Using integration by parts,

$$\delta \mathfrak{G} = \sum_{i=1}^{n} m_i \dot{x}_i^T \delta x_i |_{t_0}^{t_f} - \frac{1}{2} \mathrm{tr}[\eta_i S(J_i \Omega_i)]|_{t_0}^{t_f} + \int_{t_0}^{t_f} -m_i \ddot{x}_i^T \delta x_i$$
$$+ \frac{1}{2} \mathrm{tr}[\eta_i S(J_i \dot{\Omega}_i)] \, dt + \int_{t_0}^{t_f} -\frac{\partial U^T}{\partial x_i} \delta x_i \, dt$$
$$+ \frac{1}{2} \mathrm{tr} \left[\eta_i \left\{ S(\Omega_i \times J_i \Omega_i) + 2 R_i^T \frac{\partial U}{\partial R_i} \right\} \right] dt.$$

Using the fact that δx_i and η_i vanish at t_0 and t_f, the first two terms of the above equation vanish. Then, $\delta \mathfrak{G}$ is given by

$$\delta \mathfrak{G} = \sum_{i=1}^{n} \int_{t_0}^{t_f} -\delta x_i^T \left\{ m_i \ddot{x}_i + \frac{\partial U}{\partial x_i} \right\}$$
$$+ \frac{1}{2} \mathrm{tr} \left[\eta_i \left\{ S(J_i \dot{\Omega}_i + \Omega_i \times J_i \Omega_i) + 2 R_i^T \frac{\partial U}{\partial R_i} \right\} \right] dt.$$

From Hamilton's principle, $\delta \mathfrak{G}$ should be zero for all possible variations $\delta x_i : [t_0, t_f] \mapsto \mathbb{R}^3$ and $\eta_i : [t_0, t_f] \mapsto \mathfrak{so}(3)$. Therefore, the expression in the first brace should be zero. Furthermore, since η_i is skew symmetric, we have by (7), that the expression in the second brace should be symmetric. Then, we obtain the continuous equations of motion as

$$m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i},$$
$$S(J_i \dot{\Omega}_i + \Omega_i \times J_i \Omega_i) + 2 R_i^T \frac{\partial U}{\partial R_i}$$
$$= S(J_i \dot{\Omega}_i + \Omega_i \times J_i \Omega_i)^T + 2 \frac{\partial U^T}{\partial R_i} R. \tag{22}$$

Using (8), we can simplify (22) to be

$$S(J_i \dot{\Omega}_i + \Omega_i \times J_i \Omega_i) = \frac{\partial U^T}{\partial R_i} R - R_i^T \frac{\partial U}{\partial R_i}.$$

Note that the right hand side expression in the above equation is also skew symmetric. Then, the moment due to the gravitational potential on the ith body, $M_i \in \mathbb{R}^3$ is given by $S(M_i) = \frac{\partial U^T}{\partial R_i} R_i - R_i^T \frac{\partial U}{\partial R_i}$. This moment M_i can be expressed more explicitly as the following computation shows:

$$S(M_i) = \frac{\partial U^T}{\partial R_i} R_i - R_i^T \frac{\partial U}{\partial R_i}$$
$$= \begin{bmatrix} u_{r_{i_1}}^T & u_{r_{i_2}}^T & u_{r_{i_3}}^T \end{bmatrix} \begin{bmatrix} r_{i_1} \\ r_{i_2} \\ r_{i_3} \end{bmatrix} - \begin{bmatrix} r_{i_1}^T & r_{i_2}^T & r_{i_3}^T \end{bmatrix} \begin{bmatrix} u_{r_{i_1}} \\ u_{r_{i_2}} \\ u_{r_{i_3}} \end{bmatrix}$$
$$= (u_{r_{i_1}}^T r_{i_1} - r_{i_1}^T u_{r_{i_1}}) + (u_{r_{i_2}}^T r_{i_2} - r_{i_2}^T u_{r_{i_2}}) + (u_{r_{i_3}}^T r_{i_3} - r_{i_3}^T u_{r_{i_3}}),$$

where $r_{i_p}, u_{r_{i_p}} \in \mathbb{R}^{1\times 3}$ are the pth row vectors of R_i and $\frac{\partial U}{\partial R_i}$, respectively. Substituting $x = r_{i_p}^{\mathrm{T}}$, $y = u_{r_{i_p}}^{\mathrm{T}}$ into (9), we obtain

$$S(M_i) = S(r_{i_1} \times u_{r_{i_1}}) + S(r_{i_2} \times u_{r_{i_2}}) + S(r_{i_3} \times u_{r_{i_3}})$$
$$= S(r_{i_1} \times u_{r_{i_1}} + r_{i_2} \times u_{r_{i_2}} + r_{i_3} \times u_{r_{i_3}}), \tag{23}$$

Then, the gravitational moment M_i is given by

$$M_i = r_{i_1} \times u_{r_{i_1}} + r_{i_2} \times u_{r_{i_2}} + r_{i_3} \times u_{r_{i_3}}. \tag{24}$$

In summary, *the continuous equations of motion for the full body problem, in Lagrangian form*, can be written for $i \in (1, 2, \ldots, n)$ as

$$\dot{v}_i = -\frac{1}{m_i}\frac{\partial U}{\partial x_i}, \tag{25}$$

$$J_i\dot{\Omega}_i + \Omega_i \times J_i\Omega_i = M_i, \tag{26}$$

$$\dot{x}_i = v_i, \tag{27}$$

$$\dot{R}_i = R_i S(\Omega_i), \tag{28}$$

where the translational velocity $v_i \in \mathbb{R}^3$ is defined as $v_i = \dot{x}_i$.

3.1.2. Equations of motion: Hamiltonian form

We denote the linear and angular momentum of the ith body $\mathscr{B}_i$ by $\gamma_i \in \mathbb{R}^3$, and $\Pi_i \in \mathbb{R}^3$, respectively. They are related to the linear and angular velocities by $\gamma_i = m_i v_i$, and $\Pi_i = J_i\Omega_i$. Then, the equations of motion can be rewritten in terms of the momenta variables. *The continuous equations of motion for the full body problem, in Hamiltonian form,* can be written for $i \in (1, 2, \ldots, n)$ as

$$\dot{\gamma}_i = -\frac{\partial U}{\partial x_i}, \tag{29}$$

$$\dot{\Pi}_i + \Omega_i \times \Pi_i = M_i, \tag{30}$$

$$\dot{x}_i = \frac{\gamma_i}{m_i}, \tag{31}$$

$$\dot{R}_i = R_i S(\Omega_i). \tag{32}$$

3.2. Relative frame

The motion of the full rigid bodies depends only on the relative positions and the relative attitudes of the bodies. This is a consequence of the property that the gravitational potential can be expressed using only these relative variables. Physically, this is related to the fact that the total linear momentum and the total angular momentum about the mass center of the bodies are conserved. Mathematically, the Lagrangian is invariant under a left action of an element of SE(3). So, it is natural to express the equations of motion in one of the body fixed frame. In this section, the equations of motion for the full two body problem are derived in the relative frame. This result can be readily generalized to the n body problem.

Reduction of variables: In [2], the reduction is carried out in stages, by first reducing position variables in $\mathbb{R}^3$, and then reducing attitude variables in SO(3). This is equivalent to directly reducing the position and the attitude vari-

ables in SE(3) in a single step, which is a result that can be explained by the general theory of Lagrangian reduction by stages [24]. The reduced position and the reduced attitude variables are the relative position and the relative attitude of the first body with respect to the second body. In other words, the variables are reduced by applying the inverse of $(R_2, x_2) \in \mathrm{SE}(3)$ given by $(R_2^{\mathrm{T}}, -R_2^{\mathrm{T}}x_2) \in \mathrm{SE}(3)$, to the following homogeneous transformations:

$$\begin{bmatrix} R_2^{\mathrm{T}} & -R_2^{\mathrm{T}}x_2 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} R_1 & x_1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} R_2 & x_2 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} R_2^{\mathrm{T}}R_1 & R_2^{\mathrm{T}}(x_1 - x_2) \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} R_2^{\mathrm{T}}R_2 & R_2^{\mathrm{T}}(x_2 - x_2) \\ 0 & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} R_2^{\mathrm{T}}R_1 & R_2^{\mathrm{T}}(x_1 - x_2) \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} I_{3\times 3} & 0 \\ 0 & 1 \end{bmatrix} \right). \tag{33}$$

This motivates the definition of the reduced variables as

$$X = R_2^{\mathrm{T}}(x_1 - x_2), \tag{34}$$

$$R = R_2^{\mathrm{T}}R_1, \tag{35}$$

where $X \in \mathbb{R}^3$ is the relative position of the first body with respect to the second body expressed in the second body fixed frame, and $R \in \mathrm{SO}(3)$ is the relative attitude of the first body with respect to the second body. The corresponding linear and angular velocities are also defined as

$$V = R_2^{\mathrm{T}}(\dot{x}_1 - \dot{x}_2), \tag{36}$$

$$\Omega = R\Omega_1, \tag{37}$$

where $V \in \mathbb{R}^3$ represents the relative velocity of the first body with respect to the second body in the second body fixed frame, and $\Omega \in \mathbb{R}^3$ is the angular velocity of the first body expressed in the second body fixed frame. Here, the capital letters denote variables expressed in the second body fixed frame.

For convenience, we denote the inertial position and the inertial velocity of the second body, expressed in the second body fixed frame by $X_2, V_2 \in \mathbb{R}^3$

$$X_2 = R_2^{\mathrm{T}}x_2, \tag{38}$$

$$V_2 = R_2^{\mathrm{T}}\dot{x}_2. \tag{39}$$

Reduced lagrangian: The equations of motion in the relative frame are derived in the same way used to derive the equations in the inertial frame. We first construct a reduced Lagrangian. The reduced Lagrangian l is obtained by expressing the original Lagrangian (15) in terms of the reduced variables. The kinetic energy is given by

$$T_1 + T_2 = \frac{1}{2}m_1\|\dot{x}_1\|^2 + \frac{1}{2}m_2\|\dot{x}_2\|^2 + \frac{1}{2}\mathrm{tr}[S(\Omega_1)J_{d_1}S(\Omega_1)^{\mathrm{T}}]$$

$$\quad + \frac{1}{2}\mathrm{tr}[S(\Omega_2)J_{d_2}S(\Omega_2)^{\mathrm{T}}]$$

$$= \frac{1}{2}m_1\|V + V_2\|^2 + \frac{1}{2}m_2\|V_2\|^2$$

$$\quad + \frac{1}{2}\mathrm{tr}[S(\Omega)J_{d_R}S(\Omega)^{\mathrm{T}}] + \frac{1}{2}\mathrm{tr}[S(\Omega_2)J_{d_2}S(\Omega_2)^{\mathrm{T}}],$$

where (10) is used, and $J_{d_R} \in \mathbb{R}^{3\times3}$ is defined as $J_{d_R} = RJ_{d_1}R^T$, which is an expression of the nonstandard moment of inertia matrix of the first body with respect to the second body fixed frame. Note that J_{d_R} is not a constant matrix. Using (10), it can be shown that J_{d_R} also satisfies a property similar to (13), namely

$$S(J_R\Omega) = S(\Omega)J_{d_R} + J_{d_R}S(\Omega), \tag{40}$$

where $J_R = RJ_1R^T \in \mathbb{R}^{3\times3}$ is the standard moment of inertia matrix of the first body with respect to the second body fixed frame.

Using (14), the gravitational potential U of the full two bodies is given by

$$U(x_1, x_2, R_1, R_2) = -\int_{\mathscr{B}_1}\int_{\mathscr{B}_2} \frac{Gdm_1dm_2}{\|x_1 + R_1\rho_1 - x_2 - R_2\rho_2\|},$$

and it is invariant under an action of an element of SE(3). Therefore, the gravitational potential can be written as a function of the relative variables only. By applying the inverse of $(R_2, x_2) \in$ SE(3) as given in (33), we obtain

$$\begin{aligned} U(x_1, x_2, R_1, R_2) &= U(R_2^T(x_1 - x_2), 0, R_2^TR_1, I_{3\times3}) \\ &= -\int_{\mathscr{B}_1}\int_{\mathscr{B}_2} \frac{Gdm_1dm_2}{\|R_2^T(x_1 - x_2) + R_2^TR_1\rho_1 - I_{3\times3}\rho_2\|} \\ &= -\int_{\mathscr{B}_1}\int_{\mathscr{B}_2} \frac{Gdm_1dm_2}{\|X + R\rho_1 - \rho_2\|} \triangleq U(X, R). \end{aligned}$$

Here we abuse notation slightly by using the same letter U to denote the gravitational potential as a function of the relative variables.

Then, the reduced Lagrangian l is given by

$$\begin{aligned} l(R, X, \Omega, V, \Omega_2, V_2) &= \frac{1}{2}m_1\|V + V_2\|^2 + \frac{1}{2}m_2\|V_2\|^2 \\ &+ \frac{1}{2}\operatorname{tr}[S(\Omega)J_{d_R}S(\Omega)^T] \\ &+ \frac{1}{2}\operatorname{tr}[S(\Omega_2)J_{d_2}S(\Omega_2)^T] - U(X, R). \end{aligned} \tag{41}$$

Variations of reduced variables: The variations of the reduced variables must be restricted to those that can arise from the variations of the original variables. For example, the variation of the relative attitude R is given by

$$\delta R = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} R_2^{\epsilon T}R_1^{\epsilon} = \delta R_2^TR_1 + R_2^T\delta R_1.$$

Substituting (16) into the above equation,

$$\delta R = -\eta_2R_2^TR_1 + R_2^TR_1\eta_1 = -\eta_2R + R\eta,$$

where a reduced variation $\eta \in \mathfrak{so}(3)$ is defined as $\eta = R\eta_1R^T$. The variations of other reduced variables can be obtained in a similar way. The detailed derivations are given in A.1, and we summarize the results as follows:

$$\delta R = \eta R - \eta_2R, \tag{42}$$
$$\delta X = \chi - \eta_2X, \tag{43}$$

$$\begin{aligned} S(\delta\Omega) &= \dot{\eta} - S(\Omega)\eta + \eta S(\Omega) + S(\Omega)\eta_2 - \eta_2S(\Omega) \\ &+ S(\Omega_2)\eta - \eta S(\Omega_2), \end{aligned} \tag{44}$$
$$\delta V = \dot{\chi} + S(\Omega_2)\chi - \eta_2V, \tag{45}$$
$$S(\delta\Omega_2) = \dot{\eta}_2 + S(\Omega_2)\eta_2 - \eta_2S(\Omega_2), \tag{46}$$
$$\delta V_2 = \dot{\chi}_2 + S(\Omega_2)\chi_2 - \eta_2V_2, \tag{47}$$

where $\chi, \chi_2 : [t_0, t_f] \mapsto \mathbb{R}^3$ and $\eta, \eta_2 : [t_0, t_f] \mapsto \mathfrak{so}(3)$ are variations that vanish at the end points. These Lie group variations are the key elements required to obtain the equations of motion in the relative frame.

3.2.1. Equations of motion: Lagrangian form

The reduced equations of motion can be computed from the reduced Lagrangian using the reduced Hamilton's principle. By taking the variation of the reduced Lagrangian (41) using the constrained variations given by (42)–(47), we can obtain the equations of motion in the relative frame.

Following a similar process to the derivation of $\delta T_i, \delta U$ as in (18) and (19), the variation of the reduced Lagrangian δl can be obtained as

$$\begin{aligned} \delta l &= \dot{\chi}^T[m_1(V + V_2)] - \chi^T[m_1\Omega_2 \times (V + V_2)] \\ &+ \dot{\chi}_2^T[m_1(V + V_2) + m_2V_2] \\ &- \chi_2^T[m_1\Omega_2 \times (V + V_2) + m_2\Omega_2 \times V_2] \\ &+ \frac{1}{2}\operatorname{tr}[-\dot{\eta}S(J_R\Omega) + \eta S(\Omega_2 \times J_R\Omega)] \\ &+ \frac{1}{2}\operatorname{tr}[-\dot{\eta}_2S(J_2\Omega_2) + \eta_2S(\Omega_2 \times J_2\Omega_2)] - \chi^T\frac{\partial U}{\partial X} \\ &+ \operatorname{tr}\left[\eta_2X\frac{\partial U^T}{\partial X}\right] + \operatorname{tr}\left[\eta_2R\frac{\partial U^T}{\partial R} - \eta R\frac{\partial U^T}{\partial R}\right], \end{aligned} \tag{48}$$

where we used the identities (9), (13) and (40), and the constrained variations (42)–(47).

The action integral in terms of the reduced Lagrangian is

$$\mathfrak{G} = \int_{t_0}^{t_f} l(R, X, \Omega, V, \Omega_2, V_2)\,\mathrm{d}t. \tag{49}$$

Using integration by parts together with the fact that χ, χ_2, η and η_2 vanish at t_0 and t_f, the variation of the action integral can be expressed from (48) as (50).

$$\begin{aligned} \delta\mathfrak{G} &= -\int_{t_0}^{t_f}\chi^T\left\{m_1(\dot{V} + \dot{V}_2) + m_1\Omega_2 \times (V + V_2) + \frac{\partial U}{\partial X}\right\}\mathrm{d}t \\ &- \int_{t_0}^{t_f}\chi_2^T\{m_1(\dot{V} + \dot{V}_2) + m_2\dot{V}_2 \\ &+ m_1\Omega_2 \times (V + V_2) + m_2\Omega_2 \times V_2\}\,\mathrm{d}t \\ &+ \frac{1}{2}\int_{t_0}^{t_f}\operatorname{tr}\left[\eta\left\{S((J_R\dot{\Omega}) + \Omega_2 \times J_R\Omega) - 2R\frac{\partial U^T}{\partial R}\right\}\right]\mathrm{d}t \\ &+ \frac{1}{2}\int_{t_0}^{t_f}\operatorname{tr}\left[\eta_2\left\{S(J_2\dot{\Omega}_2 + \Omega_2 \times J_2\Omega_2) + 2X\frac{\partial U^T}{\partial X} + 2R\frac{\partial U^T}{\partial R}\right\}\right]\mathrm{d}t. \end{aligned} \tag{50}$$

From the reduced Hamilton's principle, $\delta\mathfrak{G} = 0$ for all possible variations $\chi, \chi_2 : [t_0, t_f] \mapsto \mathbb{R}^3$ and $\eta, \eta_2 : [t_0, t_f] \mapsto \mathfrak{so}(3)$

that vanish at t_0 and t_f. Therefore, in (50), the expressions in the first two braces should be zero and the expressions in the last two braces should be symmetric since η, η_2 are skew symmetric. Then, we obtain the following equations of motion:

$$m_1(\dot{V} + \dot{V}_2) + m_1\Omega_2 \times (V + V_2) = -\frac{\partial U}{\partial X}, \tag{51}$$

$$m_2\dot{V}_2 + m_2\Omega_2 \times V_2 = \frac{\partial U}{\partial X}, \tag{52}$$

$$S((J_R\dot{\Omega}) + \Omega_2 \times J_R\Omega) = -S(M),$$

$$S(J_2\dot{\Omega}_2 + \Omega_2 \times J_2\Omega_2) = \frac{\partial U}{\partial X}X^{\mathrm{T}} - X\frac{\partial U^{\mathrm{T}}}{\partial X} + S(M), \tag{53}$$

where $M \in \mathbb{R}^3$ is defined by the relation $S(M) = \frac{\partial U}{\partial R}R^{\mathrm{T}} - R\frac{\partial U^{\mathrm{T}}}{\partial R}$. By a procedure analogous to the derivation of (23), M can be written as

$$M = r_1 \times u_{r_1} + r_2 \times u_{r_2} + r_3 \times u_{r_3}, \tag{54}$$

where $r_p, u_{r_p} \in \mathbb{R}^3$ are the pth column vectors of R and $\frac{\partial U}{\partial R}$, respectively.

Eq. (51) can be simplified using (52) as

$$\dot{V} + \Omega_2 \times V = -\frac{m_1 + m_2}{m_1 m_2}\frac{\partial U}{\partial X}.$$

For reconstruction of the motion of the second body, it is natural to express the motion of the second body in the inertial frame. Since $\dot{V}_2 = \dot{R}_2^{\mathrm{T}}\dot{x}_2 + R_2^{\mathrm{T}}\ddot{x}_2 = -S(\Omega_2)V + R_2^{\mathrm{T}}\dot{v}_2$, (52) can be written as

$$m_2 R_2^{\mathrm{T}}\dot{v}_2 = \frac{\partial U}{\partial X}.$$

Eq. (53) can be simplified using the property $\frac{\partial U}{\partial X}X^{\mathrm{T}} - X\frac{\partial U^{\mathrm{T}}}{\partial X} = S(X \times \frac{\partial U}{\partial X})$ from (9). The kinematics equations for $\dot{R}$ and $\dot{X}$ can be derived in a similar way.

In summary, *the continuous equations of relative motion for the full two body problem*, *in Lagrangian form*, can be written as

$$\dot{V} + \Omega_2 \times V = -\frac{1}{m}\frac{\partial U}{\partial X}, \tag{55}$$

$$(J_R\dot{\Omega}) + \Omega_2 \times J_R\Omega = -M, \tag{56}$$

$$J_2\dot{\Omega}_2 + \Omega_2 \times J_2\Omega_2 = X \times \frac{\partial U}{\partial X} + M, \tag{57}$$

$$\dot{X} + \Omega_2 \times X = V, \tag{58}$$

$$\dot{R} = S(\Omega)R - S(\Omega_2)R, \tag{59}$$

where $m = \frac{m_1 m_2}{m_1 + m_2}$. The following equations can be used for reconstruction of the motion of the second body in the inertial frame:

$$\dot{v}_2 = \frac{1}{m_2}R_2\frac{\partial U}{\partial X}, \tag{60}$$

$$\dot{x}_2 = v_2, \tag{61}$$

$$\dot{R}_2 = R_2 S(\Omega_2). \tag{62}$$

These equations are equivalent to those given in [2]. However, (60) is not given in [2]. (55)–(62) give a complete set of equations for the reduced dynamics and reconstruction. Furthermore, they are derived systematically in the context of geometric mechanics using proper variational formulas given in (42)–(47). This result can be readily generalized for n bodies.

3.2.2. Equations of motion: Hamiltonian form

Define the linear momenta $\Gamma, \gamma_2 \in \mathbb{R}^3$, and the angular momenta $\Pi, \Pi_2 \in \mathbb{R}^3$ as

$$\Gamma = mV,$$
$$\gamma_2 = mv_2,$$
$$\Pi = J_R\Omega = RJ_1\Omega_1,$$
$$\Pi_2 = J_2\Omega_2.$$

Then, the equations of motion can be rewritten in terms of these momenta variables. *The continuous equations of relative motion for the full two body problem*, *in Hamiltonian form*, can be written as

$$\dot{\Gamma} + \Omega_2 \times \Gamma = -\frac{\partial U}{\partial X}, \tag{63}$$

$$\dot{\Pi} + \Omega_2 \times \Pi = -M, \tag{64}$$

$$\dot{\Pi}_2 + \Omega_2 \times \Pi_2 = X \times \frac{\partial U}{\partial X} + M, \tag{65}$$

$$\dot{X} + \Omega_2 \times X = \frac{\Gamma}{m}, \tag{66}$$

$$\dot{R} = S(\Omega)R - S(\Omega_2)R, \tag{67}$$

where $m = \frac{m_1 m_2}{m_1 + m_2}$. The following equations can be used to reconstruct the motion of the second body in the inertial frame:

$$\dot{\gamma}_2 = R_2\frac{\partial U}{\partial X}, \tag{68}$$

$$\dot{x}_2 = \frac{\gamma_2}{m_2}, \tag{69}$$

$$\dot{R}_2 = R_2 S(\Omega_2). \tag{70}$$

4. Lie group variational integrators

A variational integrator discretizes Hamilton's principle rather than the continuous equations of motion. Taking variations of the discretization of the action integral leads to the discrete Euler–Lagrange or discrete Hamilton's equations. The discrete Euler–Lagrange equations can be interpreted as a discrete Lagrangian map that updates the variables in the configuration space, which are the positions and the attitudes of the bodies. A discrete Legendre transformation relates the configuration variables with the linear and angular momenta variables, and yields a discrete Hamiltonian map, which is equivalent to the discrete Lagrangian map.

In this section, we derive both a Lagrangian and Hamiltonian form of variational integrators for the full body

problem in inertial and relative frames. The second level subscript k denotes the value of variables at $t = kh + t_0$ for an integration step size $h \in \mathbb{R}$ and an integer k. The integer N satisfies $t_f = kN + t_0$, so N is the number of time-steps of length h to go from the initial time t_0 to the final time t_f.

4.1. Inertial frame

Discrete Lagrangian: In continuous time, the structure of the kinematics Eqs. (28), (59) and (62) ensure that R_i, R and R_2 evolve on SO(3) automatically. Here, we introduce a new variable $F_{i_k} \in \mathrm{SO}(3)$ defined such that $R_{i_{k+1}} = R_{i_k} F_{i_k}$, i.e.

$$F_{i_k} = R_{i_k}^{\mathrm{T}} R_{i_{k+1}}. \tag{71}$$

Thus, F_{i_k} represents the relative attitude between two integration steps, and by requiring that $F_{i_k} \in \mathrm{SO}(3)$, we guarantee that R_{i_k} evolves in SO(3).

Using the kinematic equation $\dot{R}_i = R_i S(\Omega_i)$, the skew-symmetric matrix $S(\Omega_k)$ can be approximated as

$$S(\Omega_{i_k}) = R_{i_k}^{\mathrm{T}} \dot{R}_{i_k} \approx R_{i_k}^{\mathrm{T}} \frac{R_{i_{k+1}} - R_{i_k}}{h} = \frac{1}{h}(F_{i_k} - I_{3\times3}). \tag{72}$$

The velocity, $\dot{x}_{i_k}$ can be approximated simply by $(x_{i_{k+1}} - x_{i_k})/h$. Using these approximations of the angular and linear velocity, the kinetic energy of the ith body given in (12) can be approximated as

$$
\begin{aligned}
T_i(\dot{x}_i, \Omega_i) &\approx T_i\left(\frac{1}{h}(x_{i_{k+1}} - x_{i_k}), \frac{1}{h}(F_{i_k} - I_{3\times3})\right) \\
&= \frac{1}{2h^2} m_i \|x_{i_{k+1}} - x_{i_k}\|^2 \\
&\quad + \frac{1}{2h^2} \mathrm{tr}[(F_{i_k} - I_{3\times3})J_{d_i}(F_{i_k} - I_{3\times3})^{\mathrm{T}}] \\
&= \frac{1}{2h^2} m_i \|x_{i_{k+1}} - x_{i_k}\|^2 + \frac{1}{h^2} \mathrm{tr}[(I_{3\times3} - F_{i_k})J_{d_i}],
\end{aligned}
$$

where (5) is used. A discrete Lagrangian $L_d(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{R}_k, \mathbf{F}_k)$ is constructed such that it approximates a segment of the action integral (21),

$$
\begin{aligned}
L_d &= \frac{h}{2} L\left(\mathbf{x}_k, \frac{1}{h}(\mathbf{x}_{k+1} - \mathbf{x}_k), \mathbf{R}_k, \frac{1}{h}(\mathbf{F}_k - \mathbf{I})\right) \\
&\quad + \frac{h}{2} L\left(\mathbf{x}_{k+1}, \frac{1}{h}(\mathbf{x}_{k+1} - \mathbf{x}_k), \mathbf{R}_{k+1}, \frac{1}{h}(\mathbf{F}_k - \mathbf{I})\right) \\
&= \sum_{i=1}^n \frac{1}{2h} m_i \|x_{i_{k+1}} - x_{i_k}\|^2 + \frac{1}{h} \mathrm{tr}[(I_{3\times3} - F_{i_k})J_{d_i}] \\
&\quad - \frac{h}{2} U(\mathbf{x}_k, \mathbf{R}_k) - \frac{h}{2} U(\mathbf{x}_{k+1}, \mathbf{R}_{k+1}), \tag{73}
\end{aligned}
$$

where $\mathbf{x}_k \in (\mathbb{R}^3)^n$, $\mathbf{R}_k \in \mathrm{SO}(3)^n$, and $\mathbf{F}_k \in (\mathbb{R}^3)^n$, and $\mathbf{I} \in (\mathbb{R}^{3\times3})^n$ are defined as $\mathbf{x}_k = (x_{1_k}, x_{2_k}, \ldots, x_{n_k})$, $\mathbf{R}_k = (R_{1_k}, R_{2_k}, \ldots, R_{n_k})$, $\mathbf{F}_k = (F_{1_k}, F_{2_k}, \ldots, F_{n_k})$, and $\mathbf{I} = (I_{3\times3}, I_{3\times3}, \ldots, I_{3\times3})$, respectively.

This discrete Lagrangian is self-adjoint [9], and self-adjoint numerical integration methods have even order,

so we are guaranteed that the resulting integration method is at least second-order accurate.

Variations of discrete variables: The variations of the discrete variables are chosen to respect the geometry of the configuration space SE(3). The variation of x_{i_k} is given by

$$x_{i_k}^\epsilon = x_{i_k} + \epsilon \delta x_{i_k} + \mathcal{O}(\epsilon^2),$$

where $\delta x_{i_k} \in \mathbb{R}^3$ and vanishes at $k = 0$ and $k = N$. The variation of R_{i_k} is given by

$$\delta R_{i_k} = R_{i_k} \eta_{i_k}, \tag{74}$$

where $\eta_{i_k} \in \mathfrak{so}(3)$ is a variation represented by a skew-symmetric matrix and vanishes at $k = 0$ and $k = N$. The variation of F_{i_k} can be computed from the definition $F_{i_k} = R_{i_k}^{\mathrm{T}} R_{i_{k+1}}$ to give

$$
\begin{aligned}
\delta F_{i_k} &= \delta R_{i_k}^{\mathrm{T}} R_{i_{k+1}} + R_{i_k}^{\mathrm{T}} \delta R_{i_{k+1}} \\
&= -\eta_{i_k} R_{i_k}^{\mathrm{T}} R_{i_{k+1}} + R_{i_k}^{\mathrm{T}} R_{i_{k+1}} \eta_{i_{k+1}} \\
&= -\eta_{i_k} F_{i_k} + F_{i_k} \eta_{i_{k+1}}. \tag{75}
\end{aligned}
$$

4.1.1. Discrete equations of motion: Lagrangian form

To obtain the discrete equations of motion in Lagrangian form, we compute the variation of the discrete Lagrangian from (19), (74) and (75), to give

$$
\begin{aligned}
\delta L_d &= \sum_{i=1}^n \frac{1}{h} m_i(x_{i_{k+1}} - x_{i_k})^{\mathrm{T}} (\delta x_{i_{k+1}} - \delta x_{i_k}) \\
&\quad + \frac{1}{h} \mathrm{tr}[(\eta_{i_k} F_{i_k} - F_{i_k} \eta_{i_{k+1}})J_{d_i}] \\
&\quad - \frac{h}{2} \left(\frac{\partial U_k^{\mathrm{T}}}{\partial x_{i_k}} \delta x_{i_k} + \frac{\partial U_{k+1}^{\mathrm{T}}}{\partial x_{i_{k+1}}} \delta x_{i_{k+1}}\right) \\
&\quad + \frac{h}{2} \mathrm{tr}\left[\eta_{i_k} R_{i_k}^{\mathrm{T}} \frac{\partial U_k}{\partial R_{i_k}} + \eta_{i_{k+1}} R_{i_{k+1}}^{\mathrm{T}} \frac{\partial U_{k+1}}{\partial R_{i_{k+1}}}\right], \tag{76}
\end{aligned}
$$

where $U_k = U(\mathbf{x}_k, \mathbf{R}_k)$ denotes the value of the potential at $t = kh + t_0$.

Define the action sum as

$$\mathfrak{G}_d = \sum_{k=0}^{N-1} L_d(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{R}_k, \mathbf{F}_k). \tag{77}$$

The discrete action sum $\mathfrak{G}_d$ approximates the action integral (21), because the discrete Lagrangian approximates a segment of the action integral.

Substituting (76) into (77), the variation of the action sum is given by

$$
\begin{aligned}
\delta \mathfrak{G}_d &= \sum_{k=0}^{N-1} \sum_{i=1}^n \delta x_{i_{k+1}}^{\mathrm{T}} \left\{\frac{1}{h} m_i(x_{i_{k+1}} - x_{i_k}) - \frac{h}{2} \frac{\partial U_{k+1}}{\partial x_{i_{k+1}}}\right\} \\
&\quad + \delta x_{i_k}^{\mathrm{T}} \left\{-\frac{1}{h} m_i(x_{i_{k+1}} - x_{i_k}) - \frac{h}{2} \frac{\partial U_k}{\partial x_{i_k}}\right\} \\
&\quad + \mathrm{tr}\left[\eta_{i_{k+1}} \left\{-\frac{1}{h} J_{d_i} F_{i_k} + \frac{h}{2} R_{i_{k+1}}^{\mathrm{T}} \frac{\partial U_{k+1}}{\partial R_{i_{k+1}}}\right\}\right] \\
&\quad + \mathrm{tr}\left[\eta_{i_k} \left\{\frac{1}{h} F_{i_k} J_{d_i} + \frac{h}{2} R_{i_k}^{\mathrm{T}} \frac{\partial U_k}{\partial R_{i_k}}\right\}\right].
\end{aligned}
$$

Using the fact that δx_{i_k} and η_{i_k} vanish at $k = 0$ and $k = N$, we can reindex the summation, which is the discrete analogue of integration by parts, to yield

$$\delta\mathfrak{G}_d = \sum_{k=1}^{N-1}\sum_{i=1}^{n} -\delta x_{i_k}\left\{\frac{1}{h}m_i(x_{i_{k+1}} - 2x_{i_k} + x_{i_{k-1}}) + h\frac{\partial U_k}{\partial x_{i_k}}\right\}$$
$$+ \mathrm{tr}\left[\eta_{i_k}\left\{\frac{1}{h}(F_{i_k}J_{d_i} - J_{d_i}F_{i_{k-1}}) + hR_{i_k}^{\mathrm{T}}\frac{\partial U_k}{\partial R_{i_k}}\right\}\right].$$

Hamilton's principle states that $\delta\mathfrak{G}_d$ should be zero for all possible variations $\delta x_{i_k} \in \mathbb{R}^3$ and $\eta_{i_k} \in \mathfrak{so}(3)$ that vanish at the endpoints. Therefore, the expression in the first brace should be zero, and since η_{i_k} is skew-symmetric, the expression in the second brace should be symmetric. Thus, we obtain *the discrete equations of motion for the full body problem, in Lagrangian form*, for $i \in (1, 2, \ldots, n)$ as

$$\frac{1}{h}(x_{i_{k+1}} - 2x_{i_k} + x_{i_{k-1}}) = -h\frac{\partial U_k}{\partial x_{i_k}}, \tag{78}$$

$$\frac{1}{h}(F_{i_{k+1}}J_{d_i} - J_{d_i}F_{i_{k+1}}^{\mathrm{T}} - J_{d_i}F_{i_k} + F_{i_k}^{\mathrm{T}}J_{d_i}) = hS(M_{i_{k+1}}), \tag{79}$$

$$R_{i_{k+1}} = R_{i_k}F_{i_k}, \tag{80}$$

where $M_{i_k} \in \mathbb{R}^3$ is defined in (24) as

$$M_{i_k} = r_{i_1} \times u_{ri_1} + r_{i_2} \times u_{ri_2} + r_{i_3} \times u_{ri_3}, \tag{81}$$

where $r_{i_p}, u_{ri_p} \in \mathbb{R}^{1\times3}$ are pth row vectors of R_{i_k} and $\frac{\partial U_k}{\partial R_{i_k}}$, respectively. Given the initial conditions $(x_{i_0}, R_{i_0}, x_{i_1}, R_{i_1})$, we can obtain x_{i_2} from (78). Then, F_{i_0} is computed from (80), and F_{i_1} can be obtained by solving the implicit Eq. (79). Finally, R_{i_2} is found from (80). This yields an update map $(x_{i_0}, R_{i_0}, x_{i_1}, R_{i_1}) \mapsto (x_{i_1}, R_{i_1}, x_{i_2}, R_{i_2})$, and this process can be repeated.

4.1.2. Discrete equations of motion: Hamiltonian form

As discussed above, (78)–(80) defines a discrete Lagrangian map that updates x_{i_k} and R_{i_k}. The discrete Legendre transformation given in (3) and (4) relates the configuration variables x_{i_k}, R_{i_k} and the corresponding momenta. This induces a discrete Hamiltonian map that is equivalent to the discrete Lagrangian map. The discrete Hamiltonian map is particularly convenient if the initial conditions are given in terms of the positions and momenta at the initial time $(x_{i_0}, v_{i_0}, R_{i_0}, \Omega_{i_0})$.

Before deriving the variational integrator in Hamiltonian form, consider the momenta conjugate to x_i and R_i, namely $P_{v_i} \in \mathbb{R}^3$ and $P_{\Omega_i} \in \mathbb{R}^{3\times3}$. From the definition (1), $\mathbb{F}_{v_i}L$ is obtained by taking the derivative of L, given in (15), with $\dot{x}_i$ while holding other variables fixed.

$$\delta\dot{x}_i^{\mathrm{T}}P_{v_i} = \mathbb{F}_{v_i}L(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{R}, \boldsymbol{\Omega}) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}L(\mathbf{x}, \dot{\mathbf{x}} + \epsilon\delta\dot{\mathbf{x}}_i, \mathbf{R}, \boldsymbol{\Omega})$$
$$= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}T_i(\dot{x}_i + \epsilon\delta\dot{x}_i, \Omega_i) = \delta\dot{x}_i^{\mathrm{T}}(m_i\dot{x}_i),$$

where $\delta\dot{\mathbf{x}}_i \in (\mathbb{R}^3)^n$ denotes $(0, 0, \ldots, \delta\dot{x}_i, \ldots, 0)$, and T_i is given in (12). Then, we obtain

$$P_{v_i} = m_i v_i = \gamma_i, \tag{82}$$

which is equal to the linear momentum of $\mathscr{B}_i$. Similarly,

$$\mathrm{tr}[S(\delta\Omega_i)^{\mathrm{T}}P_{\Omega_i}] = \mathbb{F}_{\Omega_i}L(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{R}, \boldsymbol{\Omega}) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}T_i(\dot{x}_i, \Omega_i + \epsilon\delta\Omega_i)$$
$$= \frac{1}{2}\mathrm{tr}[S(\delta\Omega_i)^{\mathrm{T}}S(J_i\Omega_i)],$$

where (5) and (13) are used. Now, we obtain

$$\mathrm{tr}\left[S(\delta\Omega_i)^{\mathrm{T}}\left\{P_{\Omega_i} - \frac{1}{2}S(J_i\Omega_i)\right\}\right] = 0.$$

Since $S(\Omega_i)$ is skew-symmetric, the expression in the braces should be symmetric. This implies that

$$P_{\Omega_i} - P_{\Omega_i}^{\mathrm{T}} = S(J_i\Omega_i) = S(\Pi_i). \tag{83}$$

Eqs. (82) and (83) give expressions for the momenta conjugate to x_i and R_i. Consider the discrete Legendre transformations given in (3) and (4). Then,

$$\delta x_{i_k}^{\mathrm{T}}\mathbf{D}_{x_{i,k}}L_d(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{R}_k, \mathbf{F}_k)$$
$$= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}L_d(\mathbf{x}_k + \epsilon\delta\mathbf{x}_{i_k}, \mathbf{x}_{k+1}, \mathbf{R}_k, \mathbf{F}_k)$$
$$= -\delta x_{i_k}^{\mathrm{T}}\left[\frac{1}{h}m_i(x_{i_{k+1}} - x_{i_k}) + \frac{h}{2}\frac{\partial U_k}{\partial x_{i_k}}\right], \tag{84}$$

where $\delta\mathbf{x}_{i_k} \in (\mathbb{R}^3)^n$ denotes $(0, 0, \ldots, \delta x_{i_k}, \ldots, 0)$. Therefore, we have

$$\mathbf{D}_{x_{i,k}}L_d(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{R}_k, \mathbf{F}_k) = -\frac{1}{h}m_i(x_{i_{k+1}} - x_{i_k}) - \frac{h}{2}\frac{\partial U_k}{\partial x_{i_k}}. \tag{85}$$

From the discrete Legendre transformation given in (3), $P_{v_{i,k}} = -\mathbf{D}_{x_{i,k}}L_d$. Using (82) and (85), we obtain

$$\gamma_{i_k} = \frac{1}{h}m_i(x_{i_{k+1}} - x_{i_k}) + \frac{h}{2}\frac{\partial U_k}{\partial x_{i_k}}. \tag{86}$$

Using the discrete Legendre transformation given in (4), $P_{v_{i,k+1}} = \mathbf{D}_{x_{i,k}}L_d$, we can derive the following equation similarly:

$$\gamma_{i_{k+1}} = \frac{1}{h}m_i(x_{i_{k+1}} - x_{i_k}) - \frac{h}{2}\frac{\partial U_{k+1}}{\partial x_{i_{k+1}}}. \tag{87}$$

Eqs. (86) and (87) define the variational integrator in Hamiltonian form for the translational motion. Now, consider the rotational motion. We have

$$\mathrm{tr}[\eta_{i_k}\mathbf{D}_{R_{i,k}}L_d^{\mathrm{T}}] = \mathrm{tr}\left[\eta_{i_k}\left\{\frac{1}{h}F_{i_k}J_{d_i} + \frac{h}{2}R_{i_k}^{\mathrm{T}}\frac{\partial U_k}{\partial R_{i_k}}\right\}\right], \tag{88}$$

where the right side is obtained by taking the variation of L_d with respect to R_{i_k}, while holding other variables fixed. Since η_{i_k} is skew-symmetric,

$$-\mathbf{D}_{R_{i,k}}L_d + \mathbf{D}_{R_{i,k}}L_d^{\mathrm{T}} = \frac{1}{h}(F_{i_k}J_{d_i} - J_{d_i}F_{i_k}^{\mathrm{T}}) - \frac{h}{2}S(M_{i_k}), \tag{89}$$

where $M_{i_k} \in \mathbb{R}^3$ is defined in (81).

From the discrete Legendre transformation given in (3), $P_{\Omega_{i,k}} = -\mathbf{D}_{R_{i,k}}L_d$, we obtain the following equation by using (83) and (89),

$$S(\Pi_{i_k}) = \frac{1}{h}(F_{i_k}J_{d_i} - J_{d_i}F_{i_k}^{\mathrm{T}}) - \frac{h}{2}S(M_{i_k}). \tag{90}$$

Using the discrete Legendre transformation given in (4), $P_{\Omega_{i,k+1}} = \mathbf{D}_{R_{i,k+1}}L_d$, we can obtain the following equation:

$$S(\Pi_{i_{k+1}}) = \frac{1}{h}F_{i_k}^{\mathrm{T}}(F_{i_k}J_{d_i} - J_{d_i}F_{i_k}^{\mathrm{T}})F_{i_k} + \frac{h}{2}S(M_{i_{k+1}}). \tag{91}$$

By using (10) and substituting (90), we can reduce (91) to the following equation in vector form.

$$\Pi_{i_{k+1}} = F_{i_k}^{\mathrm{T}}\Pi_{i_k} + \frac{h}{2}F_{i_k}^{\mathrm{T}}M_{i_k} + \frac{h}{2}M_{i_{k+1}}. \tag{92}$$

Eqs. (90) and (92) define the variational integrator in Hamiltonian form for the rotational motion.

In summary, using (86), (87), (90) and (92), *the discrete equations of motion for the full body problem, in Hamiltonian form,* can be written for $i \in (1, 2, \ldots, n)$ as

$$x_{i_{k+1}} = x_{i_k} + \frac{h}{m_i}\gamma_{i_k} - \frac{h^2}{2m_i}\frac{\partial U_k}{\partial x_{i_k}}, \tag{93}$$

$$\gamma_{i_{k+1}} = \gamma_{i_k} - \frac{h}{2}\frac{\partial U_k}{\partial x_{i_k}} - \frac{h}{2}\frac{\partial U_{k+1}}{\partial x_{i_{k+1}}}, \tag{94}$$

$$hS\left(\Pi_{i_k} + \frac{h}{2}M_{i_k}\right) = F_{i_k}J_{d_i} - J_{d_i}F_{i_k}^{\mathrm{T}}, \tag{95}$$

$$\Pi_{i_{k+1}} = F_{i_k}^{\mathrm{T}}\Pi_{i_k} + \frac{h}{2}F_{i_k}^{\mathrm{T}}M_{i_k} + \frac{h}{2}M_{i_{k+1}}, \tag{96}$$

$$R_{i_{k+1}} = R_{i_k}F_{i_k}. \tag{97}$$

Given $(x_{i_0}, \gamma_{i_0}, R_{i_0}, \Pi_{i_0})$, we can find x_{i_1} from (93). Solving the implicit Eq. (95) yields F_{i_0}, and R_{i_1} is computed from (97). Then, (94) and (96) gives γ_{i_1}, and Π_{i_1}. This defines the discrete Hamiltonian map, $(x_{i_0}, \gamma_{i_0}, R_{i_0}, \Pi_{i_0}) \mapsto (x_{i_1}, \gamma_{i_1}, R_{i_1}, \Pi_{i_1})$, and this process can be repeated.

4.2. Relative frame

In this section, we derive the variational integrator for the full two body problem in the relative frame by following the procedure given before. This result can be readily generalized to n bodies.

Reduction of discrete variables: The discrete reduced variables are defined in the same way as the continuous reduced variables, which are given in (34)–(39). We introduce $F_k \in \mathrm{SO}(3)$ such that $R_{k+1} = R_{2_{k+1}}^{\mathrm{T}}R_{1_{k+1}} = F_{2_k}^{\mathrm{T}}F_kR_k$. i.e.

$$F_k = R_kF_{1_k}R_k^{\mathrm{T}}. \tag{98}$$

Discrete reduced Lagrangian: The discrete reduced Lagrangian is obtained by expressing the original discrete Lagrangian given in (73) in terms of the discrete reduced variables.

From the definition of the discrete reduced variables given in (34) and (38), we have

$$x_{1_{k+1}} - x_{1_k} = R_{2_{k+1}}(X_{k+1} + X_{2_{k+1}}) - R_{2_k}(X_k + X_{2_k})$$
$$= R_{2_k}\{F_{2_k}(X_{k+1} + X_{2_{k+1}}) - (X_k + X_{2_k})\}, \tag{99}$$

$$x_{2_{k+1}} - x_{2_k} = R_{2_k}\{F_{2_k}X_{2_{k+1}} - X_{2_k}\}. \tag{100}$$

From (72), $S(\Omega_{1_k})$ and $S(\Omega_{2_k})$ are expressed as

$$S(\Omega_{1_k}) = \frac{1}{h}(F_{1_k} - I_{3\times 3})$$
$$= \frac{1}{h}R_k^{\mathrm{T}}(F_k - I_{3\times 3})R_k, \tag{101}$$

$$S(\Omega_{2_k}) = \frac{1}{h}(F_{2_k} - I_{3\times 3}). \tag{102}$$

Substituting (99)–(102) into (73), we obtain the discrete reduced Lagrangian

$$l_{d_k} = l_d(X_k, X_{k+1}, X_{2_k}, X_{2_{k+1}}, R_k, F_k, F_{2_k})$$
$$= \frac{1}{2h}m_1\|F_{2_k}(X_{k+1} + X_{2_{k+1}}) - (X_k + X_{2_k})\|^2$$
$$+ \frac{1}{2h}m_2\|F_{2_k}X_{2_{k+1}} - X_{2_k}\|^2 + \frac{1}{h}\mathrm{tr}[(I_{3\times 3} - F_k)J_{dR_k}]$$
$$+ \frac{1}{h}\mathrm{tr}[(I_{3\times 3} - F_{2_k})J_{d_2}] - \frac{h}{2}U(X_k, R_k) - \frac{h}{2}U(X_{k+1}, R_{k+1}), \tag{103}$$

where $J_{dR_k} \in \mathbb{R}^{3\times 3}$ is defined to be $J_{dR_k} = R_kJ_{d_1}R_k^{\mathrm{T}}$, which gives the nonstandard moment of inertia matrix of the first body with respect to the second body fixed frame at $t = kh + t_0$.

Variations of discrete reduced variables: The variations of the discrete reduced variables can be derived from those of the original variables. The variations of R_k, X_k, and F_{2_k} are the same as given in (42), (43), and (75), respectively. The variation of F_k is computed in (A.2).

In summary, the variations of discrete reduced variables are given by

$$\delta R_k = \eta_kR_k - \eta_{2_k}R_k, \tag{104}$$

$$\delta X_k = \chi_k - \eta_{2_k}X_k, \tag{105}$$

$$\delta F_k = -\eta_{2_k}F_k + F_{2_k}\eta_{k+1}F_{2_k}^{\mathrm{T}}F_k + F_k(-\eta_k + \eta_{2_k}), \tag{106}$$

$$\delta X_{2_k} = \chi_{2_k} - \eta_{2_k}X_{2_k}, \tag{107}$$

$$\delta F_{2_k} = -\eta_{2_k}F_{2_k} + F_{2_k}\eta_{2_{k+1}}. \tag{108}$$

These Lie group variations are the main elements required to derive the variational integrator equations.

4.2.1. Discrete equations of motion: Lagrangian form

As before, we can obtain the discrete equations of motion in Lagrangian form by computing the variation of the discrete reduced Lagrangian which, by using (104)–(108), is given as (109).

$$\delta l_{d_k} = \frac{1}{h}\chi_{k+1}^{\mathrm{T}}[m_1(X_{k+1}+X_{2_{k+1}})-m_1F_{2_k}^{\mathrm{T}}(X_k+X_{2_k})]$$

$$+\frac{1}{h}\chi_k^{\mathrm{T}}[m_1(X_k+X_{2_k})-m_1F_{2_k}(X_{k+1}+X_{2_{k+1}})]$$

$$+\frac{1}{h}\chi_{2_{k+1}}^{\mathrm{T}}[m_1(X_{k+1}+X_{2_{k+1}})-m_1F_{2_k}^{\mathrm{T}}(X_k+X_{2_k})$$

$$+m_2X_{2_{k+1}}-m_2F_{2_k}^{\mathrm{T}}X_{2_k}]+\frac{1}{h}\chi_{2_k}^{\mathrm{T}}[m_1(X_k+X_{2_k})$$

$$-m_1F_{2_k}(X_{k+1}+X_{2_{k+1}})+m_2X_{2_k}-m_2F_{2_k}X_{2_{k+1}}]$$

$$-\frac{1}{h}\mathrm{tr}[\eta_{k+1}F_{2_k}^{\mathrm{T}}F_kJ_{dR_k}F_{2_k}]+\frac{1}{h}\mathrm{tr}[\eta_kF_kJ_{dR_k}]$$

$$-\frac{1}{h}\mathrm{tr}[\eta_{2_{k+1}}J_{d_2}F_{2_k}]+\frac{1}{h}\mathrm{tr}[\eta_{2_k}F_{2_k}J_{d_2}]$$

$$-\frac{h}{2}\chi_k^{\mathrm{T}}\frac{\partial U_k}{\partial X_k}+\frac{h}{2}\mathrm{tr}\left[\eta_{2_k}X_k\frac{\partial U_k^{\mathrm{T}}}{\partial X_k}\right]-\frac{h}{2}\chi_{k+1}^{\mathrm{T}}\frac{\partial U_{k+1}}{\partial X_{k+1}}$$

$$+\frac{h}{2}\mathrm{tr}\left[\eta_{2_{k+1}}X_{k+1}\frac{\partial U_{k+1}^{\mathrm{T}}}{\partial X_{k+1}}\right]+\frac{h}{2}\mathrm{tr}\left[\eta_{2_k}R_k\frac{\partial U_k^{\mathrm{T}}}{\partial R_k}-\eta_kR_k\frac{\partial U_k^{\mathrm{T}}}{\partial R_k}\right]$$

$$+\frac{h}{2}\mathrm{tr}\left[\eta_{2_{k+1}}R_{k+1}\frac{\partial U_{k+1}^{\mathrm{T}}}{\partial R_{k+1}}-\eta_{k+1}R_{k+1}\frac{\partial U_{k+1}^{\mathrm{T}}}{\partial R_{k+1}}\right]. \tag{109}$$

The action sum expressed in terms of the discrete reduced Lagrangian has the form

$$\mathfrak{G}_d = \sum_{k=0}^{N-1} l_d(X_k,X_{k+1},X_{2_k},X_{2_{k+1}},R_k,F_k,F_{2_k}). \tag{110}$$

The discrete action sum $\mathfrak{G}_d$ approximates the action integral (49), because the discrete Lagrangian approximates a piece of the integral. Using the fact that the variations $\chi_k,\chi_{2_k},\eta_k,\eta_{2_k}$ vanish at $k=0$ and $k=N$, the variation of the discrete action sum can be expressed as (111).

$$\delta\mathfrak{G}_d = \sum_{k=1}^{N-1}\frac{1}{h}\chi_k^{\mathrm{T}}\left\{-m_1F_{2_{k-1}}^{\mathrm{T}}(X_{k-1}+X_{2_{k-1}})+2m_1(X_k+X_{2_k})\right.$$

$$\left.-m_1F_{2_k}(X_{k+1}+X_{2_{k+1}})-h^2\frac{\partial U_k}{\partial X_k}\right\}$$

$$+\sum_{k=1}^{N-1}\frac{1}{h}\chi_{2_k}^{\mathrm{T}}\left\{-m_1F_{2_{k-1}}^{\mathrm{T}}(X_{k-1}+X_{2_{k-1}})+2m_1(X_k+X_{2_k})\right.$$

$$\left.-m_1F_{2_k}(X_{k+1}+X_{2_{k+1}})-m_2F_{2_{k-1}}^{\mathrm{T}}X_{2_{k-1}}+2m_2X_{2_k}-m_2F_{2_k}X_{2_{k+1}}\right\}$$

$$+\sum_{k=1}^{N-1}\mathrm{tr}\left[\eta_k\left\{\frac{1}{h}(-F_{2_{k-1}}^{\mathrm{T}}F_{k-1}R_{k-1}J_{d_1}R_{k-1}^{\mathrm{T}}F_{2_{k-1}}+F_kR_kJ_{d_1}R_k^{\mathrm{T}})-hR_k\frac{\partial U_k^{\mathrm{T}}}{\partial R_k}\right\}\right]$$

$$+\sum_{k=1}^{N-1}\mathrm{tr}\left[\eta_{2_k}\left\{\frac{1}{h}(-J_{d_2}F_{2_{k-1}}+F_{2_k}J_{d_2})+hX_k\frac{\partial U_k^{\mathrm{T}}}{\partial X_k}+hR_k\frac{\partial U_k^{\mathrm{T}}}{\partial R_k}\right\}\right]. \tag{111}$$

From Hamilton's principle, $\delta\mathfrak{G}_d$ should be zero for all possible variations $\chi_k,\chi_{2_k}\in\mathbb{R}^3$ and $\eta_k,\eta_{2_k}\in\mathfrak{so}(3)$ which vanish at the endpoints. Therefore, in (111), the expressions in the first two braces should be zero, and the expressions in the last two braces should be symmetric since η_k,η_{2_k} are skew-symmetric. After some simplification, we obtain *the discrete equations of relative motion for the full two body problem, in Lagrangian form*, as

$$F_{2_k}X_{k+1}-2X_k+F_{2_{k-1}}^{\mathrm{T}}X_{k-1} = -\frac{h^2}{m}\frac{\partial U_k}{\partial X_k}, \tag{112}$$

$$F_{k+1}J_{dR_{k+1}}-J_{dR_{k+1}}F_{k+1}^{\mathrm{T}}$$
$$= F_{2_k}^{\mathrm{T}}(F_kJ_{dR_k}-J_{dR_k}F_k^{\mathrm{T}})F_{2_k}-h^2S(M_{k+1}), \tag{113}$$

$$F_{2_{k+1}}J_{d_2}-J_{d_2}F_{2_{k+1}}^{\mathrm{T}} = F_{2_k}^{\mathrm{T}}(F_{2_k}J_{d_2}-J_{d_2}F_{2_k}^{\mathrm{T}})F_{2_k}$$
$$+h^2X_{k+1}\times\frac{\partial U_{k+1}}{\partial X_{k+1}}+h^2S(M_{k+1}), \tag{114}$$

$$R_{k+1} = F_{2_k}^{\mathrm{T}}F_kR_k, \tag{115}$$

$$R_{2_{k+1}} = R_{2_k}F_{2_k}. \tag{116}$$

It is natural to express equations of motion for the second body in the inertial frame.

$$x_{2_{k+1}}-2x_{2_k}+x_{2_{k-1}} = \frac{h^2}{m_2}R_k\frac{\partial U_k}{\partial X_k}. \tag{117}$$

Given $(X_0,R_0,R_{2_0},X_1,R_1,R_{2_1})$, we can determine F_0 and F_{2_0} from (115) and (116). Solving the implicit equations (113) and (114) gives F_1 and F_{2_1}. Then X_2, R_2 and R_{2_2} are found from (112), (115) and (116), respectively. This yields the discrete Lagrangian map $(X_0,R_0,R_{2_0},X_1,R_1,R_{2_1})\mapsto(X_1,R_1,R_{2_1},X_2,R_2,R_{2_2})$ and this process can be repeated. We can separately reconstruct x_{2_k} using (117).

4.2.2. Discrete equations of motion: Hamiltonian form

Using the discrete Legendre transformation, we can obtain the Hamiltonian map, in terms of reduced variables, that is equivalent to the Lagrangian map given in (112)–(117). We will only sketch the procedure as it is analogous to the approach of the previous section. First, we find expressions for the conjugate momenta variables corresponding to (82) and (83). We compute the discrete Legendre transformation by taking the variation of the discrete reduced Lagrangian as in (84) and (88). Then, we obtain the discrete equations of motion in Hamiltonian form using (3) and (4).

The discrete equations of relative motion for the full two body problem, in Hamiltonian form, can be written as

$$X_{k+1} = F_{2_k}^{\mathrm{T}}\left(X_k+h\frac{\Gamma_k}{m}-\frac{h^2}{2m}\frac{\partial U_k}{\partial X_k}\right), \tag{118}$$

$$\Gamma_{k+1} = F_{2_k}^{\mathrm{T}}\left(\Gamma_k-\frac{h}{2}\frac{\partial U_k}{\partial X_k}\right)-\frac{h}{2}\frac{\partial U_{k+1}}{\partial X_{k+1}}, \tag{119}$$

$$\Pi_{k+1} = F_{2_k}^{\mathrm{T}}\left(\Pi_k-\frac{h}{2}M_k\right)-\frac{h}{2}M_{k+1}, \tag{120}$$

$$\Pi_{2_{k+1}} = F_{2_k}^{\mathrm{T}}\left(\Pi_{2_k}+\frac{h}{2}X_k\times\frac{\partial U_k}{\partial X_k}+\frac{h}{2}M_k^{\mathrm{T}}\right)$$
$$+\frac{h}{2}X_{k+1}\times\frac{\partial U_{k+1}}{\partial X_{k+1}}+\frac{h}{2}M_{k+1}, \tag{121}$$

$$R_{k+1} = F_{2_k}^{\mathrm{T}}F_kR_k, \tag{122}$$

$$hS\left(\Pi_k-\frac{h}{2}M_k\right) = F_kJ_{dR_k}-J_{dR_k}F_k^{\mathrm{T}}, \tag{123}$$

$$hS\left(\Pi_{2_k}+\frac{h}{2}X_k\times\frac{\partial U}{\partial X_k}+\frac{h}{2}M_k\right) = F_{2_k}J_{d_2}-J_{d_2}F_{2_k}^{\mathrm{T}}. \tag{124}$$

It is natural to express equations of motion for the second body in the inertial frame for reconstruction:

$$x_{2_{k+1}} = x_{2_k} + h\frac{\gamma_{2_k}}{m_2} + \frac{h^2}{2m_2}R_k\frac{\partial U_k}{\partial X_k}, \tag{125}$$

$$\gamma_{2_{k+1}} = \gamma_{2_k} + \frac{h}{2}R_k\frac{\partial U_k}{\partial X_k} + \frac{h}{2}R_{k+1}\frac{\partial U_{k+1}}{\partial X_{k+1}}, \tag{126}$$

$$R_{2_{k+1}} = R_{2_k}F_{2_k}. \tag{127}$$

Given $(R_0, X_0, \Pi_0, \Gamma_0, \Pi_{2_0})$, we can determine F_0 and F_{2_0} by solving the implicit Eqs. (123) and (124). Then, X_1 and R_1 are found from (118) and (122), respectively. After that, we can compute Γ_1, Π_1, and Π_{2_1} from (119)–(121). This yields a discrete Hamiltonian map $(R_0, X_0, \Pi_0, \Gamma_0, \Pi_{2_0}) \mapsto (R_1, X_1, \Pi_1, \Gamma_1, \Pi_{2_1})$, and this process can be repeated. x_{2_k}, γ_{2_k} and R_{2_k} can be updated separately using (125)–(127), respectively, for reconstruction.

4.3. Numerical considerations

Properties of the variational integrators: Variational integrators exhibit a discrete analogue of Noether's theorem [10], and symmetries of the discrete Lagrangian result in conservation of the corresponding momentum maps. Our choice of discrete Lagrangian is such that it inherits the symmetries of the continuous Lagrangian. Therefore, all the conserved momenta in the continuous dynamics are preserved by the discrete dynamics.

The proposed variational integrators are expressed in terms of Lie group computations [12]. During each integration step, $F_{i_k} \in \mathrm{SO}(3)$ is obtained by solving an implicit equation, and R_{i_k} is updated by multiplication with F_{i_k}. Since SO(3) is closed under matrix multiplication, the attitude matrix $R_{i_{k+1}}$ remains in SO(3). We make this more explicit in Section 4.4 by expressing F_{i_k} as the exponential function of an element of the Lie algebra $\mathfrak{so}(3)$.

An adjoint integration method is the inverse map of the original method with reversed time-step. An integration method is called self-adjoint or symmetric if it is identical to its adjoint; a self-adjoint method always has even order. Our discrete Lagrangian is chosen to be self-adjoint, and therefore the corresponding variational integrators are second-order accurate.

Higher-order methods: While the numerical methods we present in this paper are second-order, it is possible to apply the symmetric composition methods, introduced in [25], to construct higher-order versions of the Lie group variational integrators introduced here. Given a basic numerical method represented by the flow map Φ_h, the composition method is obtained by applying the basic method using different step sizes,

$$\Psi_h = \Phi_{\lambda_s h} \circ \cdots \circ \Phi_{\lambda_1 h},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathbb{R}$. In particular, the Yoshida symmetric composition method for composing a symmetric method of order 2 into a symmetric method of order 4 is obtained when $s = 3$, and

$$\lambda_1 = \lambda_3 = \frac{1}{2 - 2^{1/3}}, \qquad \lambda_2 = -\frac{2^{1/3}}{2 - 2^{1/3}}.$$

Alternatively, by adopting the formalism of higher-order Lie group variational integrators introduced in [21] in conjunction with the Rodrigues formula, one can directly construct higher-order generalizations of the Lie group methods presented here.

Reduction of orthogonality loss due to roundoff error: In the Lie group variational integrators, the numerical solution is made to automatically remain on the rotation group by requiring that the numerical solution is updated by matrix multiplication with the exponential of a skew symmetric matrix.

Since the exponential of the skew symmetric matrix is orthogonal to machine precision, the numerical solution will only deviate from orthogonality due to the accumulation of roundoff error in the matrix multiplication, and this orthogonality loss grows linearly with the number of time-steps taken.

One possible method of addressing this issue is to use the Baker–Campbell–Hausdorff (BCH) formula to track the updates purely at the level of skew symmetric matrices (the Lie algebra). This allows us to find a matrix $C(t)$, such that,

$$\exp(tA)\exp(tB) = \exp C(t).$$

This matrix $C(t)$ satisfies the following differential equation,

$$\dot{C} = A + B + \frac{1}{2}[A - B, C] + \sum_{k \geqslant 2}\frac{B_k}{k!}\mathrm{ad}_C^k(A + B),$$

with initial value $C(0) = 0$, and where B_k denotes the Bernoulli numbers, and $\mathrm{ad}_C A = [C, A] = CA - AC$.

The problem with this approach is that the matrix $C(t)$ is not readily computable for arbitrary A and B, and in practice, the series is truncated, and the differential equation is solved numerically.

An error is introduced in truncating the series, and numerical errors are introduced in numerically integrating the differential equations. Consequently, while the BCH formula could be used solely at the reconstruction stage to ensure that the numerical attitude always remains in the rotation group to machine precision, the truncation error would destroy the symplecticity and momentum preserving properties of the numerical scheme.

However, by combining the BCH formula with the Rodrigues formula in constructing the discrete variational principle, it might be possible to construct a Lie group variational integrator that tracks the reconstructed trajectory on the rotation group at the level of a curve in the Lie algebra, while retaining its structure-preservation properties.

4.4. Computational approach

The structure of the discrete equations of motion given in (79), (95), (113), (114), (123), and (124) suggests

a specific computational approach. For a given $g \in \mathbb{R}^3$, we have to solve the following Lyapunov-like equation to find $F \in SO(3)$ at each integration step.

$$FJ_d - J_dF^{\mathrm{T}} = S(g). \tag{128}$$

We now introduce an iterative approach to solve (128) numerically. An element of a Lie group can be expressed as the exponential of an element of its Lie algebra, so $F \in SO(3)$ can be expressed as an exponential of $S(f) \in \mathfrak{so}(3)$ for some vector $f \in \mathbb{R}^3$. The exponential can be written in closed form, using Rodrigues' formula,

$$F = e^{S(f)} = I_{3\times3} + \frac{\sin\|f\|}{\|f\|}S(f) + \frac{1-\cos\|f\|}{\|f\|^2}S(f)^2. \tag{129}$$

Substituting (129) into (128), we obtain

$$S(g) = \frac{\sin\|f\|}{\|f\|}S(Jf) + \frac{1-\cos\|f\|}{\|f\|^2}S(f \times Jf),$$

where (9) and (13) are used. Thus, (128) is converted into the equivalent vector equation $g = G(f)$, where $G : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is

$$G(f) = \frac{\sin\|f\|}{\|f\|}Jf + \frac{1-\cos\|f\|}{\|f\|^2}f \times Jf.$$

We use the Newton method to solve $g = G(f)$, which gives the iteration

$$f_{i+1} = f_i + \nabla G(f_i)^{-1}(g - G(f_i)). \tag{130}$$

We iterate until $\|g - G(f_i)\| < \epsilon$ for a small tolerance $\epsilon > 0$. The Jacobian $\nabla G(f)$ in (130) can be expressed as

$$\nabla G(f) = \frac{\cos\|f\|\|f\| - \sin\|f\|}{\|f\|^3}Jff^{\mathrm{T}} + \frac{\sin\|f\|}{\|f\|}J$$
$$+ \frac{\sin\|f\|\|f\| - 2(1-\cos\|f\|)}{\|f\|^4}(f \times Jf)f^{\mathrm{T}}$$
$$+ \frac{1-\cos\|f\|}{\|f\|^2}\{-S(Jf) + S(f)J\}.$$

Numerical simulations show that 3 or 4 iterations are sufficient to achieve a tolerance of $\epsilon = 10^{-15}$.

5. Numerical simulations

The variational integrator in Hamiltonian form given in (118)–(127) is used to simulate the dynamics of two simple dumbbell bodies acting under their mutual gravity.

5.1. Full body problem defined by two dumbbell bodies

Each dumbbell model consists of two equal rigid spheres and a massless rod as shown in Fig. 3. The gravitational potential of the two dumbbell models is given by

$$U(X, R) = -\sum_{p,q=1}^{2} \frac{Gm_1m_2/4}{\|X + \rho_{2_p} + R\rho_{1_q}\|},$$
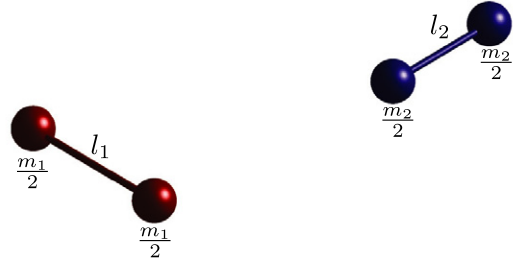
Fig. 3. Dumbbell model of the full body problem.

where G is the universal gravitational constant, $m_i \in \mathbb{R}$ is the total mass of the ith dumbbell, and $\rho_{i_p} \in \mathbb{R}^3$ is a vector from the origin of the body fixed frame to the pth sphere of the ith dumbbell in the ith body fixed frame. The vectors $\rho_{i_1} = [l_i/2, 0, 0]^{\mathrm{T}}$, $\rho_{i_2} = -\rho_{i_1}$, where l_i is the length between the two spheres.

Normalization: Mass, length and time dimensions are normalized as follows:

$$\bar{m}_i = \frac{m_i}{m},$$
$$\overline{X}_i = \frac{X_i}{l},$$
$$\bar{t} = \sqrt{\frac{G(m_1 + m_2)}{l^3}}t,$$

where $m = \frac{m_1m_2}{m_1+m_2}$, and l is chosen as the initial horizontal distance between the center of mass of the two dumbbells. The time is normalized so that the orbital period is of order unity. Over-bars denote normalized variables. We can express the equations of motion in terms of the normalized variables. For example, (55) can be written as

$$\overline{V}' + \overline{\Omega}_2 \times \overline{V} = -\frac{\partial\overline{U}}{\partial\overline{X}},$$

where $'$ denotes a derivative with respect to $\bar{t}$. The normalized gravitational potential and its partial derivatives are given by

$$\overline{U} = -\frac{1}{4}\sum_{p,q=1}^{2}\frac{1}{\|\overline{X} + \bar{\rho}_{2_p} + R\bar{\rho}_{1_q}\|},$$
$$\frac{\partial\overline{U}}{\partial\overline{X}} = \frac{1}{4}\sum_{p,q=1}^{2}\frac{\overline{X} + \bar{\rho}_{2_p} + R\bar{\rho}_{1_q}}{\|\overline{X} + \bar{\rho}_{2_p} + R\bar{\rho}_{1_q}\|^3},$$
$$\frac{\partial\overline{U}}{\partial R} = \frac{1}{4}\sum_{p,q=1}^{2}\frac{(\overline{X} + \bar{\rho}_{2_p})\bar{\rho}_{1_q}^{\mathrm{T}}}{\|\overline{X} + \bar{\rho}_{2_p} + R\bar{\rho}_{1_q}\|^3}.$$

Conserved quantities: The total energy E is conserved:

$$E = \frac{1}{2}m_1\|V + V_2\|^2 + \frac{1}{2}m_2\|V_2\|^2 + \frac{1}{2}\mathrm{tr}[S(\Omega)J_{d_R}S(\Omega)^{\mathrm{T}}]$$
$$+ \frac{1}{2}\mathrm{tr}[S(\Omega_2)J_{d_2}S(\Omega_2)^{\mathrm{T}}] + U(X, R).$$

The total linear momentum $\gamma_T \in \mathbb{R}^3$, and the total angular momentum about the mass center of the system $\pi_T \in \mathbb{R}^3$, in the inertial frame, are also conserved:

$$\gamma_T = R_2\{m_1(V + V_2) + m_2 V_2\},$$
$$\pi_T = R_2\{mX \times V + J_R\Omega + J_2\Omega_2\}.$$

5.2. Simulation results

The properties of the two dumbbell bodies are chosen to be

$$\bar{m}_1 = 1.5, \quad \bar{l}_1 = 0.25, \quad \overline{J}_1 = \mathrm{diag}[0.0004, 0.0238, 0.0238],$$
$$\bar{m}_2 = 3, \quad \bar{l}_2 = 0.5, \quad \overline{J}_2 = \mathrm{diag}[0.0030, 0.1905, 0.1905].$$

The mass and length of the second dumbbell are twice that of the first dumbbell. The initial conditions are chosen such that the total linear momentum in the inertial frame is zero and the total energy is positive.

$$\overline{X}_0 = [1, 0, 0.3], \quad \overline{V}_0 = [0, 1, 0],$$
$$\overline{\Omega}_{1_0} = [0, 0, 9], \quad R_0 = I_{3\times 3},$$
$$\bar{x}_{2_0} = [-0.33, 0, -0.1], \quad \bar{v}_{2_0} = [0, -0.33, 0],$$
$$\overline{\Omega}_{2_0} = [0, 0, 0], \quad R_{2_0} = I_{3\times 3}.$$

Simulation results obtained using the Lie group variational integrator are given in Figs. 4 and 5. Fig. 4 shows the trajectory of the two dumbbells in the inertial frame. Fig. 5(a) shows the evolution of the normalized energy, where the upper figure gives the history of the translational kinetic energy and the rotational kinetic energy, and the lower figure shows the interchange between the total kinetic energy and the gravitational potential energy. Fig. 5(b) shows the evolution of the theoretically conserved quantities, where the upper figure is the history of the total energy, and the lower figure is the error in the rotation matrix.

Initially, the first dumbbell rotates around the vertical e_3 axis, and the second dumbbell does not rotate. Since the angular velocity of the first dumbbell is relatively large, the rotational kinetic energy initially exceeds the translational kinetic energy. As the two dumbbells orbit around
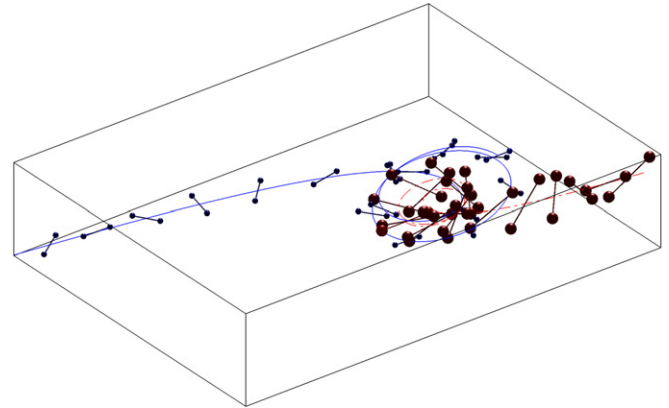

Fig. 4. Trajectory in the inertial frame.

each other, the second dumbbell starts to rotate, the rotational kinetic energy increases, and the translational kinetic energy decreases slightly for about 6 normalized units of time. At 9 units of time, the distance between the two dumbbells reaches its minimal separation, and the potential energy is transformed into kinetic energy, especially translational kinetic energy. After that, two dumbbells continue to move apart, and the translational energy and the rotational energy equalize. (A simple animation of this motion can be found at http://www.umich.edu/~tylee.) This shows some of the interesting dynamics that the full body problem can exhibit. The non-trivial interchange between rotational kinetic energy, translational kinetic energy, and potential energy may yield complicated motions that cannot be observed in the classical two body problem.

The Lie group variational integrator preserves the total energy and the geometry of the configuration space. The maximum deviation of the total energy is 2.6966×10^{-7}, and the maximum value of the rotation matrix error $\|I - R^T R\|$ is 2.8657×10^{-13}.

As a comparison, Fig. 6 shows simulation results obtained by numerically integrating the continuous equations of motion (63)–(70) using a standard Runge–Kutta
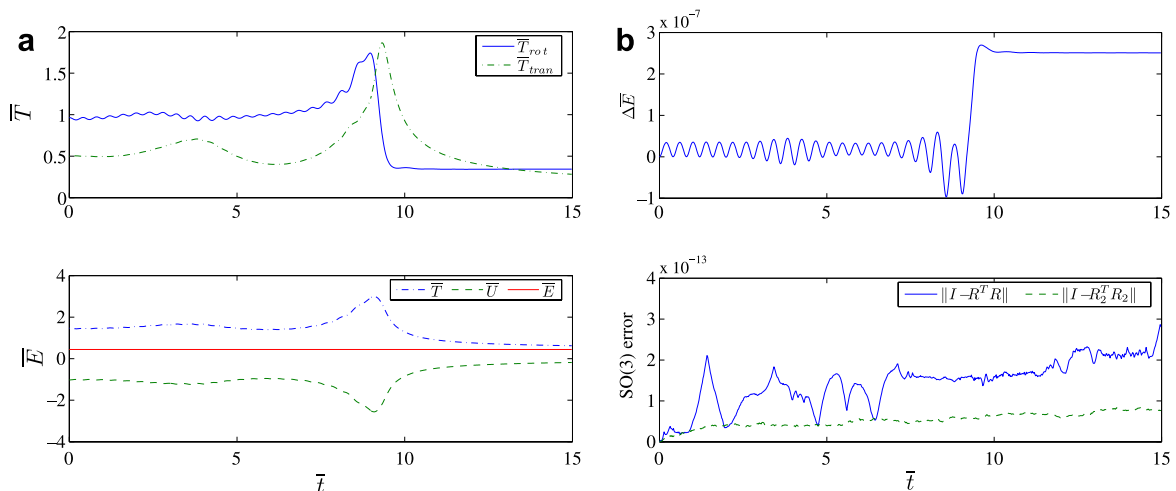

Fig. 5. Lie group variational integrator: (a) interchange of energy and (b) conserved quantities.
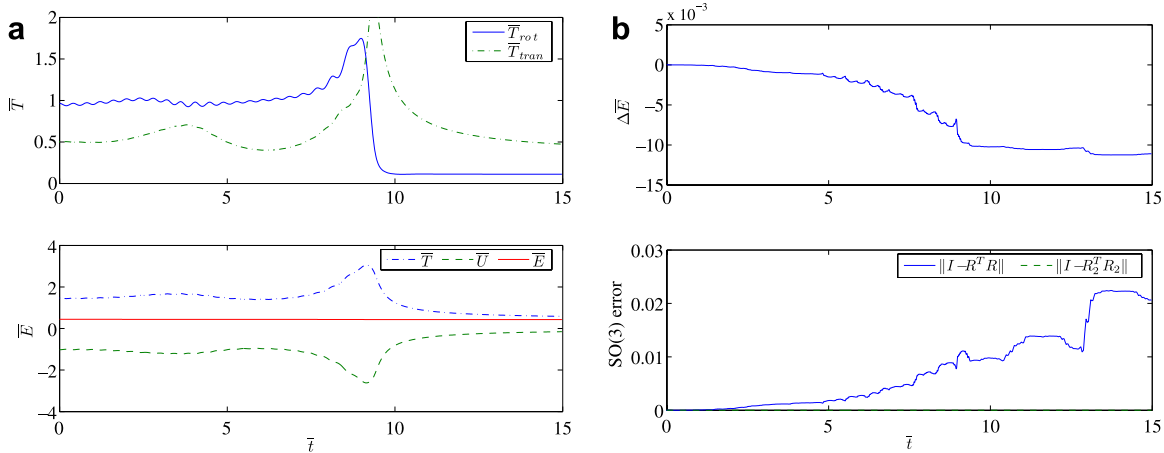
Fig. 6. Runge–Kutta method: (a) interchange of energy and (b) conserved quantities.

method. The rotational and the translational kinetic energy responses are similar to those given in Fig. 5 prior to the close encounter. However, it fails to simulate the rapid interchange of the energy near the minimal separation of the two dumbbells. The deviation of the total energy is relatively large, with a maximum deviation of 1.1246×10^{-2}. Also, the energy transfer is quite different from that given in Fig. 5(a). The Runge–Kutta method does not preserve the geometry of the configuration space, as the discrete trajectory rapidly drifts off the rotation group to give a maximum rotation matrix error of 2.2435×10^{-2}. As the gravity and momentum between the two dumbbells depend on the relative attitude, the errors in the rotation matrix limits the applicability of standard techniques to long time simulations.

An extensive computational comparison between the Lie group variational integrator to other geometric integrators such as symplectic Runge–Kutta method and Lie group method can be found in [26].

6. Conclusions

Eight different forms of the equations of motion for the full body problem are derived. The continuous equations of motion and variational integrators are derived both in the inertial and relative frames, and each set of equations of motion is expressed in both Lagrangian and Hamiltonian form. The relationships between these equations of motion are summarized in Fig. 7. This commutative cube was originally given in [22]. In the figure, dashed arrows represent discretization from the continuous systems on the left face of the cube to the discrete systems on the right face. Vertical arrows represent reduction from the full (inertial) equations on the top face to the reduced (relative) equations on the bottom face. Front and back faces represent Lagrangian and Hamiltonian forms, respectively. The corresponding equation numbers are also indicated in parentheses.

It is shown that the equations of motion for the full body problem can be derived systematically, using proper
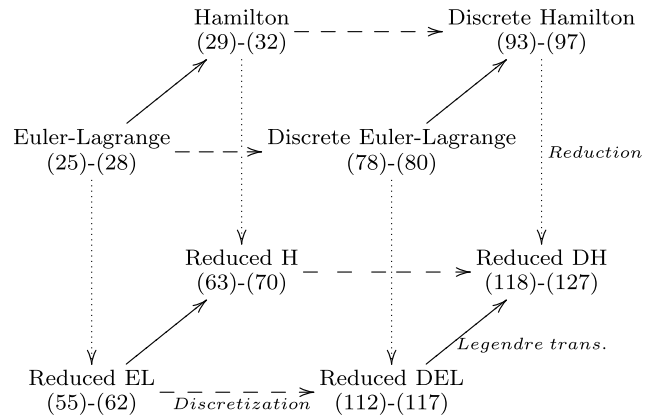

Fig. 7. Commutative cube of the equations of motion.

Lie group variations, from Hamilton's principle. The proposed variational integrators preserve the momenta and symplectic form of the continuous dynamics, exhibit good energy properties, and they also conserve the geometry of the configuration space since they are based on Lie group computations. The main contribution of this paper is the combination of variational integrators and Lie group computations, developed for the full body problem. Hence, the resulting numerical integrators conserve the first integrals as well as the geometry of the configuration space of the full body dynamics.

Appendix A. Variations of reduced variables

A.1. Continuous variables

The variations of the reduced variables given in (43)–(47) are derived in this section. The variations of the reduced variables can be obtained from the definitions of the reduced variables, and the variations of the original variables.

The variation of $X = R_2^{\mathrm{T}}(x_1 - x_2)$ is given by

$$\delta X = \delta R_2^{\mathrm{T}}(x_1 - x_2) + R_2(\delta x_1 - \delta x_2).$$

Substituting (16) into the above equation, we obtain

$$\delta X = -\eta_2 R_2^T(x_1 - x_2) + R_2(\delta x_1 - \delta x_2) = -\eta_2 X + \chi,$$

where the reduced variation $\chi : [t_0, t_f] \mapsto \mathbb{R}^3$ is defined to be $\chi = R_2(\delta x_1 - \delta x_2)$.

From the definition of $\Omega = R\Omega_1$ and (10), $S(\delta\Omega)$ is given by

$$S(\delta\Omega) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} S(R^\epsilon \Omega_1^\epsilon) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} R^\epsilon S(\Omega_1^\epsilon) R^{\epsilon T}$$
$$= \delta R S(\Omega_1) R^T + R S(\delta\Omega_1) R^T + R S(\Omega_1)\delta R^T.$$

Substituting (42) and (17) into the above equation, we obtain

$$S(\delta\Omega) = \{\eta - \eta_2\} R S(\Omega_1) R^T + R\{\dot\eta_1 + S(\Omega_1)\eta_1 - \eta_1 S(\Omega_1)\} R^T$$
$$\quad + R S(\Omega_1) R^T \{-\eta + \eta_2\}$$
$$= \{\eta - \eta_2\} S(R\Omega_1) + R\dot\eta_1 R^T + S(R\Omega_1) R\eta_1 R^T$$
$$\quad - R\eta_1 R^T S(R\Omega_1) + S(R\Omega_1)\{-\eta + \eta_2\}.$$

Since $\eta = R\eta_1 R^T$ and $\Omega = R\Omega_1$, the above equation reduces to

$$S(\delta\Omega) = -\eta_2 S(\Omega) + R\dot\eta_1 R^T + S(\Omega)\eta_2. \tag{A.1}$$

From the definition of $R = R_2^T R_1$, $\dot R$ is given by

$$\dot R = \dot R_2^T R_1 + R_2^T \dot R_1 = -S(\Omega_2)R + S(\Omega)R. \tag{A.2}$$

Then, $\dot\eta$ can be written as

$$\dot\eta = R\dot\eta_1 R^T + \dot R\eta_1 R^T + R\eta_1 \dot R^T$$
$$= R\dot\eta_1 R^T + \{S(\Omega) - S(\Omega_2)\}\eta - \eta\{S(\Omega) - S(\Omega_2)\}. \tag{A.3}$$

Substituting (A.3) into (A.1), we obtain $S(\delta\Omega)$ in terms of η, η_2 as

$$S(\delta\Omega) = \dot\eta - S(\Omega)\eta + \eta S(\Omega) + S(\Omega)\eta_2 - \eta_2 S(\Omega) + S(\Omega_2)\eta$$
$$\quad - \eta S(\Omega_2),$$

which is equivalent to (44).

The variation of $V = R_2^T(\dot x_1 - \dot x_2)$ is given by

$$\delta V = \delta R_2^T(\dot x_1 - \dot x_2) + R_2^T(\delta \dot x_1 - \delta \dot x_2)$$
$$= -\eta_2 V + R_2^T(\delta \dot x_1 - \delta \dot x_2). \tag{A.4}$$

From the definition of $\chi = R_2^T(\delta x_1 - \delta x_2)$, $\dot\chi$ is given by

$$\dot\chi = \dot R_2^T(\delta x_1 - \delta x_2) + R_2^T(\delta x_1 - \delta x_2)$$
$$= -S(\Omega_2)\chi + R_2^T(\delta \dot x_1 - \delta \dot x_2). \tag{A.5}$$

Substituting (A.5) into (A.4), we obtain

$$\delta V = -\eta_2 V + \dot\chi + S(\Omega_2)\chi,$$

which is equivalent to (45). The variation δV_2 can be derived in the same way, and $S(\delta\Omega_2)$ is given in (17).

A.2. Discrete variables

The variation of the reduced variables δF_k given in (106) is derived in this section. From (75) and (98), the variation δF_{1_k} is written as

$$\delta F_{1_k} = -\eta_{1_k} F_{1_k} + F_{1_k}\eta_{1_{k+1}}$$
$$= -R_k^T \eta_k F_k R_k + R_k^T F_k R_k R_{k+1}^T \eta_{k+1} R_{k+1},$$

where $\eta_k \in \mathfrak{so}(3)$ is defined as $\eta_k = R_k\eta_{1_k}R_k^T$. Since $F_k R_k R_{k+1}^T = F_k R_k(R_k^T F_k^T F_{2_k}) = F_{2_k}$, we have

$$\delta F_{1_k} = R_k^T(-\eta_k F_k + F_{2_k}\eta_{k+1}F_{2_k}^T F_k)R_k.$$

Then, the variation δF_k is given by

$$\delta F_k = \delta R_k F_{1_k} R_k^T + R_k \delta F_{1_k} R_k^T + R_k F_{1_k}\delta R_k^T$$
$$= -\eta_{2_k} F_k + F_{2_k}\eta_{k+1}F_{2_k}^T F_k + F_k(-\eta_k + \eta_{2_k}),$$

which is equivalent to (106).

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