MATH 170C ASSIGNMENT 4

(1) Write the theta method,

$$y_{n+1} = y_n + h[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})]$$

as a Runge–Kutta method, and express it in the form of a Butcher tableau.

(2) Derive the three-stage Runge–Kutta method that corresponds to the collocation points $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{3}{4}$, and determine its order of accuracy.

(3) Given $\theta \in [0, 1]$, find the order of the method,

$$y_{n+1} = y_n + hf(t_n + (1-\theta)h, \theta y_n + (1-\theta) y_{n+1}).$$

Remark: Note that this is not the theta method given in the first problem, rather it is a generalization of the midpoint rule, as opposed to the trapezoidal rule.

(4) Provided that f is analytic, it is possible to obtain from $y' = f(t, y)$ an expression for the second derivative of y, namely $y'' = g(t, y)$, where

$$g(t, y) = \frac{\partial f(t, y)}{\partial t} + f(t, y) \frac{\partial f(t, y)}{\partial y}.$$

Find the orders of the methods

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{1}{2} h^2 g(t_n, y_n)$$

and

$$y_{n+1} = y_n + \frac{1}{2} h[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] + \frac{1}{12} h^2 [g(t_n, y_n) - g(t_{n+1}, y_{n+1})].$$