MATH 170C ASSIGNMENT 2

(1) $\boxed{\textbf{SOURCE CODE:}}$

Write separate MATLAB functions for

- Composite Simpson's Rule: `I = Simpson(@f,a,b,n)`
- Romberg Integration: `I = Romberg(@f,a,b,n)`

(2) $\boxed{\textbf{DISCUSSION QUESTION:}}$

Use your Composite Simpson's Rule code to approximate the integral

$$I = \int_0^{10} \cos(2x)\, dx\,.$$

Obtain approximations using $n = 2^k$ for $k = 1, 2, 3, \ldots, 10$, and numerical verify that the method has $\mathcal{O}(h^4)$ rate of convergence.

For this question, turn in a table that contains 4 columns:

(a) the various h values,
(b) the approximations to I,
(c) the errors,
(d) the ratio of the previous/current error.

After you have constructed these tables, write down a brief statement explaining why the results in the table verify that the Composite Simpson's method is $O(h^4)$.

(3) $\boxed{\textbf{DISCUSSION QUESTION:}}$

Use your Romberg Integration code to approximate the same integral as in Question #3. Obtain approximations using $n = 4, 5, 6, 7, 8, 9, 10$.

For this question, turn in a table that contains 4 columns:

(a) the various n values,
(b) the approximations to I,
(c) the errors,
(d) the ratio of the previous/current error.

(4) $\boxed{\textbf{APPLICATION PROBLEM:}}$

The heat flow in a thin rod can be modeled using the **HEAT EQUATION** from classical physics:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial s^2} = 0,$$

where $u(s,t)$ is the temperature at location s along the rod and at time t. The **HEAT EQUATION** is a partial differential equation.

If the rod is sufficiently long so that the temperatures at the left and right ends of the rod do not significantly affect the overall temperature, we can assume that the rod is infinetely long.

Consider an initial temperature distribution that is given by the following function:

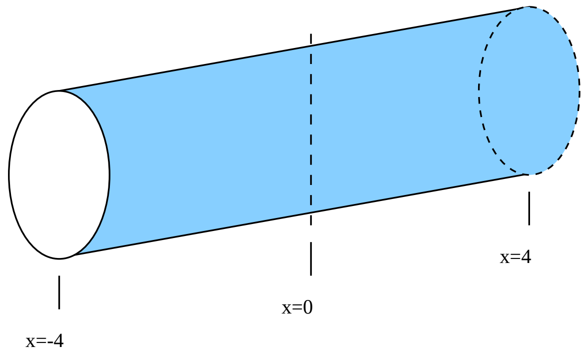

Figure 1. Application problem: heat flow in a thin rod.

$$f(s) = \begin{cases} 0.5\cos(\pi s) + 0.5 & \text{if } |s| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

One can show that the solution to the **HEAT EQUATION** with the above initial condition can be written as follows:

$$\begin{aligned} t = 0: &\qquad u(s,0) = f(s) \\ t > 0: &\qquad u(s,t) = \int_{-1}^{1} K(x,s,t)\,dx\,, \\ \text{where} &\qquad K(x,s,t) = \frac{1}{4\sqrt{\pi t}}\Big(\cos(\pi x) + 1\Big)e^{\left(\frac{-(s-x)^2}{4t}\right)}. \end{aligned}$$

(a) Verify that

$$u(s,t) = \int_{-1}^{1} \frac{1}{4\sqrt{\pi t}}\Big(\cos(\pi x) + 1\Big)e^{\left(\frac{-(s-x)^2}{4t}\right)}\,dx$$

is a solution to the heat equation. (**HINT:** Plug the above expression for $u(s,t)$ into the heat equation. Partial derivatives can be brought inside the integral.)

(b) Modify your Composite Simpson's Rule code so that it can accept `s` and `t` as parameters:

```
function u = modSimpson(f,a,b,n,s,t)
```

Inside your `modSimpson.m` code change your `feval` statements so that they also pass `s` and `t` to f:

OLD: `feval(f,x)`
NEW: `feval(f,x,s,t)`

(c) For any given value of s and t, we have to compute an integral in order to obtain an approximation to $u(s,t)$. Write a new m-file that evaluates $u(s,t)$ at all the following points using your Composite Simpson's rule:

```
s = linspace(-4,4,201);
t = linsapce(0,1,6);
```

In your Composite Simpson's method use $a = -1$, $b = 1$, $n = 200$, and $K(x, s, t)$ as the function you are integrating with respect to x.

NOTE: When $t = 0$ just use the definition of the initial condition; at $t = 0$ the integral is singular.

(d) Make a single MATLAB plot that contains the solution $u(s, t)$ at the six output times:

$$\begin{aligned} &u(s, 0) \quad \text{vs.} \quad s \\ &u(s, 0.2) \quad \text{vs.} \quad s \\ &\vdots \\ &u(s, 1) \quad \text{vs.} \quad s\,. \end{aligned}$$

(e) Explain in words what you observe about the behavior of the temperature distribution as a function of space and time.