
Problem 1 Determine all polynomials $P(x)$ of degree $n \geq 1$ such that $xP(x) = (x - n)P(x + 1)$, for every real number x .

$P(x)$ must equal $C(x - 1)(x - 2) \cdots (x - n)$ for some constant $C \neq 0$. First, we verify that these work. We have that

$$\begin{aligned} xP(x) &= x(C(x - 1)(x - 2) \cdots (x - n)) = Cx(x - 1) \cdots (x - n) \\ &= (x - n)(Cx(x - 1) \cdots (x - n + 1)) = (x - n)P(x + 1). \end{aligned}$$

Furthermore for $C \neq 0$, this is clearly degree n .

To show that these are the only solutions note that from $xP(x) = (x - n)P(x + 1)$, we see that n is a root of $xP(x)$, and therefore a root of $P(x)$. But then noting that $P(x + 1) = 0$ when $x = n - 1$, $n - 1$ must also be a root. We claim by induction that for all $k < n$ that $n - k$ must be a root of P . In particular, the base case of $k = 1$ has already been proved. Assuming that $n - k$ is a root, plugging $x = n - k - 1$ into $xP(x) = (x - n)P(x + 1)$ gives that $n - k - 1$ is also a root completing our induction.

Thus, $P(x)$ must have as roots $1, 2, \dots, n$. Thus, if it is degree n it must be $C(x - 1)(x - 2) \cdots (x - n)$ for some non-zero C .

Note: This still holds without the assumption that $\deg(P) = n$.

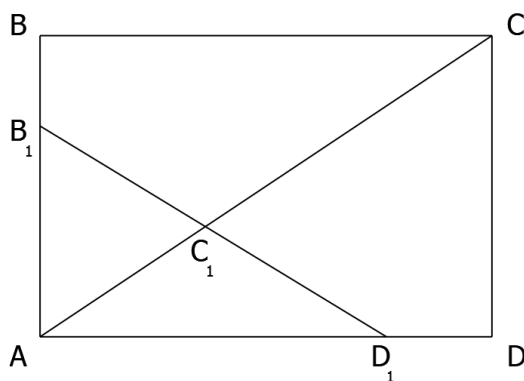
Problem 2 Consider a rectangle $ABCD$. Let α denote the angle BAC . Let B_1 and D_1 be points distinct from A on the segments AB and AD , respectively. Denote by C_1 the intersection of B_1D_1 and AC . Prove that

$$\frac{1}{|AC_1|} = \frac{\cos \alpha}{|AB_1|} + \frac{\sin \alpha}{|AD_1|}.$$

Note that $\text{area}(AB_1D_1) = \text{area}(AB_1C_1) + \text{area}(AC_1D_1)$. Note also that $\text{area}(AB_1D_1) = |AB_1||AD_1|/2$ and that $\text{area}(AB_1C_1) = |AB_1||AC_1| \sin(\alpha)/2$ and $\text{area}(AC_1D_1) = |AC_1||AD_1| \cos(\alpha)/2$. Plugging these into the previous equation gives

$$|AB_1||AD_1|/2 = |AB_1||AC_1| \sin(\alpha)/2 + |AC_1||AD_1| \cos(\alpha)/2.$$

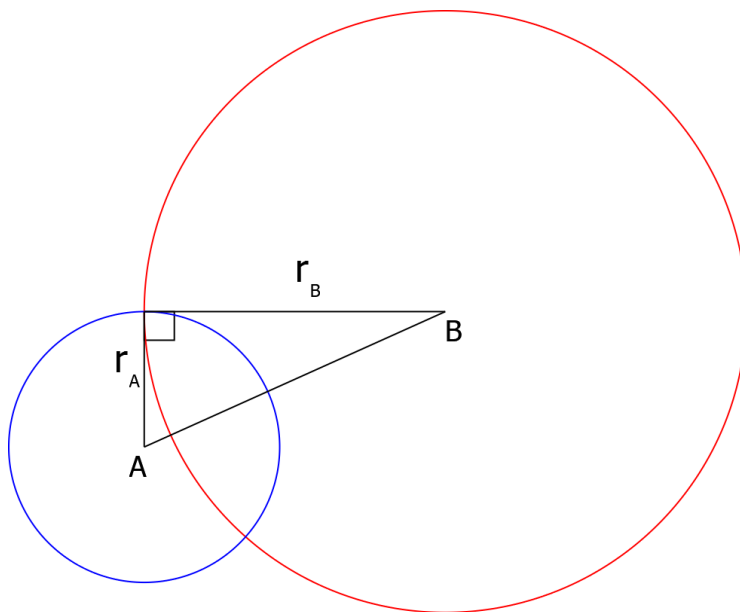
Multiplying by $2/|AB_1||AC_1||AD_1|$ yields our result.



Problem 3 Suppose that four circles are pairwise orthogonal to each other. Show that the centers of these circles form a triangle and its orthic center (the orthic center of a triangle is the intersection of its altitudes).

Let the centers of these circles be A, B, C, D . We want to show that the lines AB and CD are orthogonal. In particular, this means that D is on the altitude from C on triangle ABC . Symmetrically, it must also be on the altitudes from A and B and thus be the orthic center.

Let the circles centered at A, B, C, D have radii r_A, r_B, r_C, r_D respectively. We claim that $AB^2 = r_A^2 + r_B^2$ and the same for other pairs of vertices. This is because if we let I be the intersection of the circles about A and B , the tangents to A 's and B 's circles at I are orthogonal to AI and BI respectively. Therefore AI and BI must be orthogonal to each other, which implies that ABI is a right triangle (with right angle at I). Since the side lengths of this triangle are AB, r_A and r_B , the Pythagorean Theorem implies that $r_A^2 + r_B^2 = AB^2$.



Next, we note that

$$AC^2 + BD^2 = r_A^2 + r_B^2 + r_C^2 + r_D^2 = AD^2 + BC^2.$$

Rearranging gives

$$AC^2 - BC^2 = AD^2 - BD^2.$$

Let c and d be the base of the altitudes from C to AB and from D to AB . By the Pythagorean Theorem we have that

$$AC^2 - BC^2 = Ac^2 + cC^2 - Bc^2 - cC^2 = Ac^2 - Bc^2.$$

Similarly, this equals

$$AD^2 - BD^2 = Ad^2 - Bd^2.$$

However, for x on the line AB , $Ax^2 - Bx^2$ is monotonic in x , so if $Ac^2 - Bc^2 = Ad^2 - Bd^2$, it must hold that $c = d$, and thus that AB and CD are orthogonal.

This completes our proof.

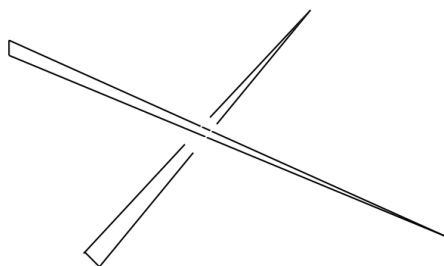
Problem 4 Let T_n be the maximum number of triangles that can be placed in three dimensional space so that no two of these triangles overlap except on corners or edges and all together there are at most n points used as vertices of these triangles. Prove that for $n \geq 3$:

$$(n - 3)^2/2 \leq T_n \leq n^2.$$

For the upper bound consider one of these n points x and consider the set of triangles with x as one of its corners. Draw a small sphere around x and consider its intersection with each triangle. Each triangle will intersect the sphere at an arc whose endpoints are given by the intersection of this sphere with the ray between x and one of the other $n - 1$ points. Furthermore, no two of these arcs will cross each other. Thus, they will produce a planar graph with at most $n - 1$ vertices and edges equal to the number of triangles in our set with a vertex at x . By Euler's Formula the number of edges of a planar graph is at most three times the number of vertices, which for us is at most $3n$. Summing this over all vertices x gives at most $3n^2$, which counts each triangle three times (once for each vertex). Thus, the number of triangles is at most n^2 .

For the lower bound, we will show that when n is even that it is possible to draw $(n - 2)(n/2) \geq (n - 2)^2/2$ many triangles. For this let $x_1, \dots, x_{n/2}$ be random points on the unit sphere. For some sufficiently small $\epsilon > 0$ let y_i be a random point at distance ϵ from x_i . We then for each i and j draw triangles $x_i y_i x_j$ and $x_i y_i y_j$. The total number of these triangles is $2(n/2)(n/2 - 1) = (n/2)(n - 2)$, and there are only n distinct vertices. We have yet to show that for ϵ small enough, that with positive probability none of these triangles have any interior intersections.

Consider the triangles $x_i y_i z_j$ and $x_k y_k z_\ell$ where z_j is either x_j or y_j and z_ℓ is either x_ℓ or y_ℓ . If i, j, k, ℓ are all distinct then all points in the first triangle are within ϵ of the line $x_i x_j$ and all points in the second within ϵ of the line $x_k x_\ell$. As two lines between random points are skew almost certainly, for ϵ small enough, these lines will likely not be within ϵ .



Next consider the case where $\{i, j\} = \{k, \ell\}$. In this case both triangles are faces of the tetrahedron $x_i y_i x_j y_j$ so unless these points are coplanar (which almost surely they are not), the faces will not overlap.

Finally, consider the case where one of i, j is one of k, ℓ . Call the overlapping index t . Project both triangles onto a plane orthogonal to $x_t y_t$. Note that this projection sends x_t and y_t to the same point p . The two triangles project to either a triangle or a segment with a vertex at p and the remaining vertex(es) within ϵ of the same x_s . However, with high probability (for ϵ small enough), the distance from p to the projection of each other

x_i is much more than $\sqrt{\epsilon}$ and the angles between the rays px_g, px_h are at least $2\sqrt{\epsilon}$ radians. If so, these two triangles cannot overlap as one will be within $\sqrt{\epsilon}$ of the ray px_g and the other close to the ray px_h for some $g \neq h$.

