Name: ID Number:

Problem 1 *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be two functions such that*

$$f(x)f(y) = [g(x+y)], \quad f(0) = 1.$$

Show that $f(x) = 1$ *for all* $x \in \mathbb{R}$*. Here* $[x]$ *denotes the integer part of* x*.*

Let $x, y \in \mathbb{R}$. We have

$$f(x+y)f(0) = [g(x+y)], \quad f(x)f(y) = [g(x+y)] \implies f(x+y)f(0) = f(x)f(y).$$

Since $f(0) = 1$, we obtain

$$f(x+y) = f(x)f(y).$$

Thus $f(nx) = f(x)^n$ for all integers $n > 0$ by induction on n. Setting $x \mapsto x/n$, we find

$$f(x) = f(x/n)^n.$$

For $y = 0$, we obtain $f(x) = [g(x)]$. Thus $f(x)$ is an integer for all $x \in \mathbb{R}$.

Fix $x \in \mathbb{R}$ and let $n > 0$ be an integer. Since

$$f(x) = f(x/n)^n$$

and $f(x/n)$ is an integer, it follows that $f(x)$ is an integer which is always an nth power. Since n is arbitrary, this implies $f(x) = 0$ or $f(x) = 1$.

If $f(x) = 0$, then

$$0 = f(x)f(2-x) = f(2) = f(1)f(1) = 1$$

yields a contradiction. Thus $f(x) = 1$ for all $x \in \mathbb{R}$.

Problem 2 *Let S be a set of 132 positive integers such that $3 \in S$ and $97 \in S$. Show that there exist three distinct elements $a, b, c \in S$ and $[a, b] \leq c$. Here, $[a, b]$ denotes the least common multiple of a and b.*

If $1 \in S$ then $(1, 3, 97)$ is a triple with the required property. Similarly if $2 \in S$, then $(2, 3, 97)$ is a triple with the required property. Thus, we may assume $1, 2 \notin S$.

Let $m = \min\{s : s \in S, s \neq 3\}$ and $M = \max\{s : s \in S\}$. Note $3 < m < M$. Also $M > 97$ since S has 132 elements which are distinct positive integers.

If $M \geq 3m$, then $(3, m, M)$ is a triple with the required property since $[3, m] \leq 3m \leq M$. Thus, we may assume $M \leq 3m - 1$ or equivalently $m \geq \frac{M+1}{3}$.

If $M \geq 291 = 3 \cdot 97$, then $(3, 97, M)$ is a triple that works. Thus, we may assume $M \leq 290$.

If there exists $s \in S \backslash \{3, M\}$ with $3|s$ then $(3, s, M)$ is a triple with the required property since $[3, s] = s < M$. Thus, we may assume all elements in $S \setminus \{3, M\}$ are not divisible by 3.

By induction on the nonnegative integers α, β, we show that the set $\{\alpha, \alpha + 1, \cdots, \beta\}$ contains $\leq \frac{2}{3}(\beta - \alpha + 2)$ numbers not divisible by 3.

Recall that S has 132 elements. Two of these elements are 3 and M. The remaining ones are in the set $\{m, \ldots, M - 1\}$ and are not divisible by 3. There are $\leq \frac{2}{3}(M - m + 1)$ elements in the latter set which are not divisible by 3. Thus

$$132 \leq 2 + \frac{2}{3}(M - m + 1).$$

Using $m \geq \frac{M+1}{3}$, we obtain

$$132 \leq 2 + \frac{2}{3}\left(M - \frac{M + 1}{3} + 1\right) \implies M > 290,$$

a contradiction.

Problem 3 *Let $n \geq 3$. Prove that the maximum number of line segments which can be drawn between the vertices of a regular n-gon in the plane without having two disjoint line segments is n.*

First we may consider all $n - 1$ lines segments containing a vertex v of a regular n-gon plus the single line segment between the vertices immediately adjacent to v in the n-gon to get n line segments without two disjoint line segments. So at least n line segments are possible without two disjoint line segments.

Now let L be any collection of $n + 1$ line segments between the vertices of a regular n-gon. For each vertex v of the n-gon contained in some line segment, let ℓ_v denote the line segment joining v to the nearest vertex of the n-gon in the clockwise direction. Remove for each vertex v the line segment ℓ_v. The number of line segments removed is at most n. Therefore there exists a remaining line segment joining a vertex u to a vertex v. Now ℓ_v and ℓ_u are disjoint line segments. Therefore at most n line segments are possible if no two are disjoint.

We conclude the maximum number of line segments is n.

Problem 4 *Prove or disprove: For any positive integer N there exists a set $S \subset \{1, 2, \ldots, N^2\}$ with $|S| = N$ so that there is no integer n that can be written as $n = x + y$ with $x, y \in S$ in more than 2023 different ways.*

The statement is true.

For a prime $p > 2$ let S_p denote the set of integers of the form $x + 2p(x^2 \pmod{p})$ where $1 \leq x \leq p$ is an integer and $(x^2 \pmod{p})$ is the unique integer between 0 and $p-1$ (inclusive) that is congruent to x^2 modulo p. We note that $S_p \subset \{1, 2, \ldots, 2p^2\}$, and $|S_p| = p$. We also claim that any integer n can be written as a sum of two elements of S_p in at most two ways.

In particular, suppose that

$$n = (x+2p(x^2 \pmod{p}))+(y+2p(y^2 \pmod{p})) = (z+2p(z^2 \pmod{p}))+(w+2p(w^2 \pmod{p})).$$

Taking both sides modulo p, we have that $x + y \equiv z + w \pmod{p}$. Furthermore, noting that $1 < x + y, z + w \leq 2p$, subtracting 1 from both sides, dividing by $2p$ and taking the floor yields that

$$(x^2 \pmod{p}) + (y^2 \pmod{p}) = (z^2 \pmod{p}) + (w^2 \pmod{p}).$$

Taking both sides modulo p yields that $x^2 + y^2 \equiv z^2 + w^2 \pmod{p}$. So if $x + y \equiv z + w \equiv a \pmod{p}$ and $x^2+y^2 \equiv z^2+w^2 \equiv b \pmod{p}$, then $xy \equiv zw \equiv (a^2-b)/2 \pmod{p}$. Therefore, $\{x, y\}$ and $\{z, w\}$ are both the set of roots modulo p of $t^2 - at + (a^2 - b)/2$. Thus, $\{x, y\} = \{z, w\}$, so n can be written as a sum of two elements of S in at most two ways.

For the full solution, note that if $N < 2023$ we can just take $S = \{1, 2, \ldots, N\}$. Otherwise, let $p > 2$ be a prime with $N/10 < p < N/5$. Let

$$S_0 = (S_p + 2p^2) \cup (S_p + 4p^2) \cup (S_p + 6p^2) \cup \ldots \cup (S_p + 20p^2),$$

where $(S_p + m)$ denotes the set of elements of the form $x + m$ with $x \in S_p$. Note that S_0 has $10p > N$ elements, each of size at most $22p^2 < N^2$. Furthermore, no integer n can be written as the sum of an element of $(S_p + m_1)$ and an integer in $(S_p + m_2)$ in more than two ways (since each such way corresponds to a way of writing $n - m_1 - m_2$ as a sum of two elements of S_p), so no integer n can be written as the sum of two elements of S in more than $2 \cdot 10^2 < 2023$ ways.

Taking a S to be any subset of S_0 of size exactly N gives a set with the desired property.