66th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 22, 2023
on the campus of the
University of California, San Diego

PART I
25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number at the top of **each page of the contest**. (Keep your ID number handy for Part II.)
2. Calculators or other electronic devices may **not** be used.
3. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.
4. Clearly indicate your answer in the bubble for each problem. Fill in the bubble, or put a clear X, or a checkmark; just make sure your selection is clear.

Scoring
$+4$ points for a correct answer.
0 points for no answer.
-1 points for an incorrect answer.

Good Luck!

There will be a 15 minute break after Part I before proceeding to Part II.

Please let your coach know if you plan to attend the Awards Banquet on Tuesday, May 2, 6:00–8:30pm.

Name: ID Number:

Problem 1 *Sonja and Talib decide to play a card game, and they each grab some cards from a deck. Sonja ends up with one third of the original deck, but Talib only has a fourth of the cards from the deck. Sonja feels bad for Talib and gives cards to Talib until they have the same number. Suppose she gave him 5 cards. How many were in the deck originally?*

◯ *(A) 52*

◯ *(B) 60*

◯ *(C) 100*

◯ *(D) 120*

◯ *(E) 212*

Problem 2 *Let T_3 be a triangle, and in general for $n \geq 4$ let T_n be obtained from T_{n-1} by adding a point inside each triangle of T_{n-1} that does not already contained any points inside it, and drawing line segments from the point to the vertices of the triangle it lies in. Then the number of triangles in T_n is*

◯ *(A)* $n-2$

◯ *(B)* $3n-8$

◯ *(C)* $3^{n-2} - 3 \cdot 2^{n-3} + 1$

◯ *(D)* 4^{n-3}

◯ *(E) None of the above*

Problem 3 *Let $A_1A_2A_3A_4$ be a rhombus of side 1 and diagonal $A_1A_3 = 1$. Outside the rhombus, construct four squares $A_1A_2M_2M_1$, $A_2A_3M_4M_3$, $A_3A_4M_6M_5$, $A_1A_4M_7M_8$. What is the area of the octagon $M_1M_2M_3M_4M_5M_6M_7M_8$?*

◯ *(A)* $4 + \frac{\sqrt{3}}{2}$

◯ *(B)* $4 + \frac{3\sqrt{3}}{4}$

◯ *(C)* $4 + \sqrt{3}$

◯ *(D)* $4 + \frac{5\sqrt{3}}{4}$

◯ *(E)* $4 + \frac{3\sqrt{3}}{2}$

Problem 4 *Let a, b, c be non-zero real numbers. Assume that $-5a^4b^3c^3$ and $7a^6b^5c^{-6}$ have the same sign. Which of the following is always correct?*

- ◯ *(A)* $a < 0$
- ◯ *(B)* $b > 0$
- ◯ *(C)* $c < 0$
- ◯ *(D)* $bc < 0$
- ◯ *(E)* $ab < 0$

Problem 5 *Find the* 2023*rd digit after the decimal point for the number x where*

$$x^2 = 0.\underbrace{4444...4}_{2023}.$$

- ◯ *(A) 1*
- ◯ *(B) 5*
- ◯ *(C) 6*
- ◯ *(D) 7*
- ◯ *(E) 8*

Problem 6 *The largest size of a subset S of $\{1, 2, 3, \ldots, 13\}$ such that all differences $a - b$ with $a, b \in S$ and $a \neq b$ are distinct is*

- ◯ *(A)* 3
- ◯ *(B)* 4
- ◯ *(C)* 5
- ◯ *(D)* 6
- ◯ *(E)* 7

Problem 7 *Let C be the surface of a cube of side length* 5*. Let R be the set of points within distance* 1 *of C. What is the surface area of R?*

- ◯ *(A)* 150
- ◯ *(B)* $150 + 30\pi$
- ◯ *(C)* $150 + 34\pi$
- ◯ *(D)* $204 + 30\pi$
- ◯ *(E)* $204 + 34\pi$

Problem 8 *The greatest common factor of* 12345 *and* 54321 *is*

◯ *(A)* 1

◯ *(B)* 3

◯ *(C)* 7

◯ *(D)* 11

◯ *(E)* 13

Problem 9 *Simplify the following expression:* $1+\frac{3}{2+\frac{3}{2+\frac{3}{2+\ddots}}}$

◯ *(A)* $\sqrt{2}$

◯ *(B)* $3/2$

◯ *(C)* 2

◯ *(D)* 3

◯ *(E)* $\sqrt{5}$

Problem 10 *The number of prime factors of* $2^{2^4}-1$ *is*

◯ *(A)* 1

◯ *(B)* 2

◯ *(C)* 3

◯ *(D)* 4

◯ *(E)* 5

Problem 11 *Let* $a,b,c,d>0$ *be positive integers. Assume that there exist infinitely many integers* n *such that* $(n+a)(n+b)$ *divides* $cn^2+cdn+d$*. The maximum possible value of* $ab^3c^5d^7$ *equals*

◯ *(A)* 1

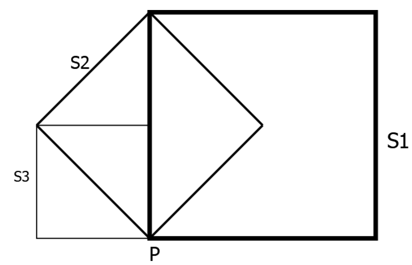
◯ *(B)* 120

◯ *(C)* 1024

◯ *(D)* 2^{18}

◯ *(E) none of the above*

Problem 12 *Let S_1 be a square with side length 1 and P as one of its vertices. Rotating S_1 forty-five degrees counter-clockwise about P and scaling it down by a factor of $\sqrt{2}$, one obtains a square S_2 with P as a vertex so that one of S_1's sides is S_2's diagonal. Rotating S_2 forty-five degrees counter-clockwise about P and scaling down, one obtains S_3 one of whose diagonals is a side of S_2 as shown below. Repeating this process, one obtains $S_4, S_5, S_6, \ldots$. What is the area of the union of the squares $S_1, S_2, S_3, \ldots$?*



- ◯ *(A)* $191/128$
- ◯ *(B)* $383/256$
- ◯ *(C)* $3/2$
- ◯ *(D)* $511/256$
- ◯ *(E)* 2

Problem 13 *How many three digit numbers are multiples of 7 and have their digits sum to 7?*

- ◯ *(A)* 3
- ◯ *(B)* 4
- ◯ *(C)* 5
- ◯ *(D)* 6
- ◯ *(E)* 7

Problem 14 *Which of the numbers below is the median?*

- ◯ *(A)* 20^{32}
- ◯ *(B)* 20^{23}
- ◯ *(C)* 32^{20}
- ◯ *(D)* 20^{2^3}
- ◯ *(E)* 23^{20}

Problem 15 *Consider the system of equations*

$$\begin{cases} 3^x + 4^x + 5^x &= 2^x 3^{x-1} y \,, \\ 3^y + 4^y + 5^y &= 2^y 3^{y-1} z \,, \\ 3^z + 4^z + 5^z &= 2^z 3^{z-1} x \,, \end{cases}$$

where x, y, z are real numbers.

How many solutions does the system have?

◯ *(A)* 0

◯ *(B)* 1

◯ *(C)* 3

◯ *(D)* 100

◯ *(E) Infinitely many*

Problem 16 *A laminar family is a collection $A_1, A_2, \ldots, A_N$ of distinct subsets of $\{1, 2, \ldots, n\}$ such that $A_i \subset A_j$ or $A_j \subset A_i$ or $A_i \cap A_j = \emptyset$ for $1 \leq i < j \leq N$. Then the maximum value of N for which there exists a laminar family of subsets of $\{1, 2, \ldots, n\}$ is*

◯ *(A)* n

◯ *(B)* $n + 1$

◯ *(C)* $2n - 1$

◯ *(D)* $2n$

◯ *(E) None of the above*

Problem 17 *Find the number of tuples $(a_1, \ldots, a_{2023}, b_1, \ldots, b_{2023})$ of 0 and 1's such that*

$$a_1 b_1 + \ldots + a_{2023} b_{2023}$$

is even.

◯ *(A)* $2^{4045} - 2^{2022}$

◯ *(B)* $2^{4045} + 2^{2022}$

◯ *(C)* $2^{4046} + 2^{2023}$

◯ *(D)* 4^{2022}

◯ *(E)* 4^{2023}

Problem 18 *Let C_1, C_2, C_3 be clubs consisting of three people each, with exactly one person p in all three clubs. In particular, $C_1 \cap C_2 = C_2 \cap C_3 = C_3 \cap C_1 = \{p\}$. If the seven people in $C_1 \cup C_2 \cup C_3$ are randomly ranked, what is the probability that p has the highest rank in at least one of the clubs C_1, C_2, C_3?*

◯ *(A)* $2/7$

◯ *(B)* $1/2$

◯ *(C)* $19/27$

◯ *(D)* $19/35$

◯ *(E)* $2/3$

Problem 19 *Let*

$$A = \frac{1}{\sqrt{10,001}} + \frac{1}{\sqrt{10,002}} + \ldots + \frac{1}{\sqrt{20,001}} \; .$$

The first digit of A before the decimal point is

◯ *(A)* 0

◯ *(B)* 1

◯ *(C)* 2

◯ *(D)* 3

◯ *(E)* 4

Problem 20 *We write $\overline{a_1 a_2 \ldots a_n}$ for the number whose decimal representation has digits $a_1, \ldots, a_n$. Consider all the solutions of the equation*

$$\sqrt{\overline{a_1 a_2 \ldots a_{n-1} a_n}} - \sqrt{\overline{a_1 a_2 \ldots a_{n-1}}} = a_n \; ,$$

where $n > 1$ is a positive integer and $a_1, a_2, \ldots, a_n$ are digits and $a_1 \neq 0$. Which of the following is true?

◯ *(A) For all solutions, $a_1 + a_2 + \ldots + a_n$ is a perfect square*

◯ *(B) There is at least one solution for which $a_1 \cdot a_2 \cdot \ldots \cdot a_n$ is a perfect square*

◯ *(C) There are exactly 2 solutions to this equation*

◯ *(D) There are exactly 3 solutions to this equation*

◯ *(E) None of the above are true*

Name: ID Number:

Problem 21 *Let u, v, w, x be vectors of length 1 in the plane and let $a, b, c, d \in \{-1, 1\}$ be chosen such that the length $\ell(u, v, w, x)$ of $au + bv + cw + dx$ is a minimum. Then the maximum value of $\ell(u, v, w, x)$ is*

- ◯ *(A)* 0
- ◯ *(B)* 1
- ◯ *(C)* $\sqrt{2}$
- ◯ *(D)* 2
- ◯ *(E)* $2\sqrt{2}$

Problem 22 *What is the remainder of $\lfloor \frac{7}{16}(17^{2023} - 1) \rfloor$ when divided by 2023?*

- ◯ *(A) 1*
- ◯ *(B) 7*
- ◯ *(C) 12*
- ◯ *(D) 126*
- ◯ *(E) 2022*

Problem 23 *The triangle ΔABC has medians BD and CE with lengths $|BD| = a$ and $|CE| = b$. What is the maximum value that the area of ΔABC can take, as a function of a and b?*

- ◯ *(A)* $\frac{4}{9}ab$
- ◯ *(B)* $\frac{2}{3}ab$
- ◯ *(C)* $\frac{2}{3}a^2 + \frac{1}{3}ab + \frac{2/3}{b}^2$
- ◯ *(D)* $\frac{1}{3}(a + b)^2$
- ◯ *(E)* $\frac{2}{3}(a + b)^2$

Problem 24 *Define a sequence of integers by $a_0 = m$ and $a_{n+1} = a_n^2 - 2$ for $n \geq 0$. For how many integers $1 \leq m \leq 1153$ is it the case that $a_{10} - 2$ is a multiple of 1153? Note that 1153 is prime.*

- ◯ *(A)* 64
- ◯ *(B)* 65
- ◯ *(C)* 128
- ◯ *(D)* 512
- ◯ *(E)* 1024

Problem 25 *Given four real numbers $0 < a < b < c < d < a + b + c$, you wish to construct the quadrilateral with side lengths a, b, c, d and the largest possible area. In this largest quadrilateral, in what order will these side lengths appear?*

- ◯ *(A) The side of length d will be opposite the side of length a*
- ◯ *(B) The side of length d will be opposite the side of length b*
- ◯ *(C) The side of length d will be opposite the side of length c*
- ◯ *(D) The answer depends on the values of a, b, c, d*
- ◯ *(E) A quadrilateral of maximum area can be constructed with any ordering of the sides*