65th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 16, 2022
on the campus of the
University of California, San Diego

PART II
4 Questions

Welcome to Part II of the contest!

Please print your Name, School, and Contest ID number:

Name ____
First   Last

School ____

Contest ID number ________________________________

Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number on **each page of the contest**.
2. Part II consists of 4 problems, each worth 25 points. These problems are "essay" style questions. You should put all work towards a solution in the space following the problem statement. You should show all work and justify your responses as best you can.
3. Scoring is based on the progress you have made in understanding and solving the problem. The clarity and elegance of your response is an important part of the scoring. You may use the back side of each sheet to continue your solution, but be sure to call the reader's attention to the back side if you use it.
4. Give this entire exam to a proctor when you have completed the test to your satisfaction.

Please let your coach know if you plan to attend the Awards Banquet on Wednesday, April 27, 6:00–8:30pm.

Problem 1 *Solve the equation below for x:*

$$\cos(\pi \log_3(x+6))\cos(\pi \log_3(x-2)) = 1.$$

Solution: The only solution is $x = 3$.

Note that since $|\cos(x)| \leq 1$, this can have a solution only if $|\cos(\pi \log_3(x+6))| = |\cos(\pi \log_3(x-2))| = 1$. This happens only if $\log_3(x+6)$ and $\log_3(x-2)$ are both integers. Thus, it must be the case that $x + 6 = 3^n$ and $x - 2 = 3^m$ for some integers n and m. Subtracting, we find that $3^n - 3^m = 8$. The fact that $3^n > 8$ implies that $n \geq 2$. If $n = 2$, we must have $m = 0$, which gives $x = 3$, which is a solution. If $n > 2$, then either $m \geq n$ in which case $3^n - 3^m \leq 0$ (which cannot happen), or $m \leq n - 1$, in which case $3^n - 3^m \geq (2/3)3^n \geq (2/3)27 = 18 > 8$ (which again is impossible).

Thus the only solution is $x = 3$.

Problem 2 *Let BD be a fixed line segment. Find the geometric locus (set of all points) A such that there exists an isosceles triangle ABC for which $AB = AC$ so that BD is the median of the edge AC.*

Solution: The locus is the circle of radius $(2/3)BD$ centered at the point on BD distance $BD/3$ beyond D.

Given a point A, let C' be the reflection of A across D. We note that BD will be the median of AC if and only if $C = C'$. The triangle ABC' is isosceles with $AB = AC'$ if and only if $AB = 2AD$. Thus, we need the locus of points so that $AB = 2AD$. This is a circle of Apollonius.

In particular, if we put B and D in the coordinate plane with $B = (0, 0)$ and $D = (1, 0)$, then the set of allowable points A are the points (x, y) with

$$\sqrt{x^2 + y^2} = 2\sqrt{(x-1)^2 + y^2}.$$

Squaring, we see that this is equivalent to

$$x^2 + y^2 = 4x^2 - 8x + 4 + 4y^2.$$

Rearranging, this gives

$$(x - 4/3)^2 + y^2 = 4/9.$$

This is the circle of radius $2/3$ centered at $(4/3, 0)$.

Problem 3 *Suppose we have an infinite sequence of numbers $a_0, a_1, \ldots, a_n, \ldots$, all between 0 and 1, such that for any $n \geq 0$,*

$$a_{n+2} - 2a_{n+1} + a_n \geq 0 \ .$$

Show that the sequence must be decreasing, and that $0 \leq a_n - a_{n+1} \leq \frac{1}{n+1}$ for all n.

Solution: Define the sequence of finite differences $D_n = a_{n+1} - a_n$. Notice that the equation in question is exactly equivalent to $D_{n+1} \geq D_n$, and the desired statement is to prove that $0 \geq D_n \geq -1/(n+1)$.

On the one hand, if $D_n > 0$ for some n, then $D_n \leq D_{n+1} \leq D_{n+2} \leq \ldots$. This means that for each $m \geq n$ that $a_{m+1} \geq a_m + D_n$. By induction on k this implies that $a_{n+k} \geq a_n + kD_n$. Taking $k > 1/D_n$, this is larger than 1, yielding a contradiction.

If, on the other hand, $D_n < -1/(n+1)$, we have that $-1/(n+1) > D_n \geq D_{n-1} \geq D_{n-2} \geq \ldots \geq D_0$. Thus, for $m \leq n$, $a_m - a_{m+1} = -D_m > 1/(n+1)$. Therefore, by induction on k we have that $a_{n+1-k} > a_{n+1} + k/(n+1)$. Applying this for $k = n+1$ yields a contradiction.

Problem 4 *Suppose that $z_1, z_2, \ldots, z_n$ are complex numbers so that for every integer k from 1 to n we have that $\sum_{i=1}^n z_i^k = 2n$. What is $\sum_{i=1}^n z_i^{n+1}$?*

Solution: The answer is $2n + (-1)^{n+1}(2n)\binom{2n-1}{n}$.

Thinking of the z_i as formal variables let p_k denote the polynomial $\sum_{i=1}^n z_i^k$. We claim that there is a polynomial P of degree at most $n+1$ so that $p_{n+1} = P(p_1, p_2, \ldots, p_n)$. This is a standard fact in the theory of symmetric polynomials, but we prove it below.

First, let σ_i denote the i^{th} elementary symmetric polynomial in the z's (i.e. the sum of all products of i distinct z_j's). We claim that each σ_i is a polynomial in the p_j's for $j \leq i$. We show this by starting with σ_i and repeatedly subtracting off products of the p_j's until there is nothing left. We will only use terms here total degree equal to i, so at any given stage we will have some symmetric polynomial in the z_j's that is homogenous of degree i. We consider a term with as many different z's as possible, say $z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots z_k^{a_k}$. By subtracting off an appropriate multiple of $p_{a_1} p_{a_2} p_{a_3} \cdots p_{a_k}$ we can remove this term (and the other symmetric ones) and introduce only new terms with fewer z's in them. By repeating this process, eventually we will eliminate all terms with k distinct z_j's in them, and then iterating again, we will remove all terms with $k+1$ distinct z_j's and so on until nothing is left.

Next, we claim that each p_i can be written as a polynomial in the σ_j's with each term homogeneous of degree i in the z's. We again do this by starting with p_i and subtracting off polynomials in the σ_j's until nothing is left. At each stage we will have a homogenous degree i symmetric polynomial in the z's. We now take a term with as few z's as possible (say $z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots z_k^{a_k}$ with $a_1 \geq a_2 \geq \ldots \geq a_k$). Subtracting off an appropriate multiple of $\sigma_k^{a_k} \sigma_{k-1}^{a_{k-1}-a_k} \cdots \sigma_1^{a_1-a_2}$, this eliminates that term (and the symmetric ones) while only introducing new terms with more z's in them. We can repeat this process until there is nothing left.

Thus, we can write p_{n+1} as a polynomial in the σ_i, which can be written as polynomials in the p_j for $j \leq n$. If we throw away all the terms not of homogeneous degree $n+1$ in the z's, the equality will still hold, and we will be left with a polynomial of degree at most n.

Note that if $p_1 = p_2 = \ldots = p_n = x$ for some x that $p_{n+1} = P(x, x, \ldots, x) = Q(x)$ for Q some degree at most $n+1$ polynomial. Our answer is $Q(2n)$. We will attempt to determine Q be finding its values at certain special numbers.

Note that if k of the z's are 1 and the other $n-k$ are 0 that $p_i = k$ for all i. Thus $Q(k) = k$ for $k = 0, 1, 2, \ldots, n$. This means that $Q(x) - x$ has roots at $0, 1, 2, \ldots, n$, and since it is a degree $n+1$ polynomial (and thus has at most $n+1$ roots), $Q(x) = x + C_n x(x-1)(x-2)\cdots(x-n)$ for some C_n.

To find C we note that if we take w to be a primitive $(n+1)^{st}$ root of unity and take $z_i = w^i$ then for $1 \leq k \leq n$, we have that

$$p_k = \sum_{i=1}^n z_i^k = \sum_{i=1}^n w^{ik} = \sum_{i=0}^n (w^k)^i - 1 = \frac{1 - w^{k(n+1)}}{1 - w^k} - 1 = -1.$$

On the other hand, we have that

$$p_{n+1} = \sum_{i=1}^{n} z_i^k = \sum_{i=1}^{n} 1 = n.$$

Thus, $Q(-1) = n$. So

$$n = -1 + C_n(-1)(-2)\cdots(-n-1) = -1 + C_n(-1)^{n+1}(n+1)!.$$

Therefore, $C_n = (-1)^{n+1}/n!$.

The answer we are looking for is $Q(2n)$ which is

$$Q(2n) = 2n + (-1)^{n+1}/n!(2n)(2n-1)\cdots(n) = 2n + (-1)^{n+1}(2n)\binom{2n-1}{n}.$$