65th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 16, 2022
on the campus of the
University of California, San Diego

PART I
25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number at the top of **each page of the contest**. (Keep your ID number handy for Part II.)
2. Calculators or other electronic devices may **not** be used.
3. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.
4. Clearly indicate your answer in the bubble for each problem. Fill in the bubble, or put a clear X, or a checkmark; just make sure your selection is clear.

Scoring
$+4$ points for a correct answer.
0 points for no answer.
-1 points for an incorrect answer.

Good Luck!

There will be a 15 minute break after Part I before proceeding to Part II.

Name: ID Number:

Problem 1 *An $n \times n \times n$ box is sliced with planes parallel to its faces until it consists of n^3 unit boxes. What is the minimum number of slices needed?*

◯ *(A)* n

◯ *(B)* $2n$

◯ *(C)* $3(n-1)$

◯ *(D)* $3n$

◯ *(E)* n^2

Problem 2 *A regular hexagon has side length s. Let x be the distance from the center of the hexagon to the midpoint of one of the sides. What is the area of the hexagon?*

◯ *(A)* xs

◯ *(B)* $2xs$

◯ *(C)* $3xs$

◯ *(D)* $6xs$

◯ *(E)* $12xs$

Problem 3 *Alice is older than Bob who is older than Charles. On a non-leap year there are 281 days when Alice and Bob are the same age and 124 days when Bob and Charles are the same age. For how many days in a non-leap year are Alice and Charles the same age?*

◯ *(A) 0*

◯ *(B) 40*

◯ *(C) 157*

◯ *(D) 208*

◯ *(E) 365*

Problem 4 *How many 4 digit numbers have exactly 2 distinct digits?*

◯ *(A) 540*

◯ *(B) 567*

◯ *(C) 648*

◯ *(D) 675*

◯ *(E) 720*

Problem 5 *Let a, b and c be positive real numbers. The expression* $\frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)}$ *simplifies to:*

◯ *(A)* 0

◯ *(B)* 1

◯ *(C)* $\log_a(bc) + \log_b(ac) + \log_c(ab)$

◯ *(D)* $\log_{abc}(a + b + c)$

◯ *(E) The expression does not simplify*

Problem 6 *Chess is played on an 8×8 board. A king on a given square can attack any square that is adjacent vertically, horizontally or diagonally. What is the largest number of kings that can be placed on different squares so that none of the kings is attacking any other king?*

◯ *(A)* 10

◯ *(B)* 12

◯ *(C)* 14

◯ *(D)* 16

◯ *(E)* 18

Problem 7 *The digits from 1 to 9 are randomly assembled to form a 9-digit number with each digit appearing exactly once. What is the probability that the result is not divisible by 45?*

◯ *(A)* $1/18$

◯ *(B)* $1/9$

◯ *(C)* $4/9$

◯ *(D)* $5/9$

◯ *(E)* $8/9$

Problem 8 *Define a sequence by $a_0 = 2$ and $a_n = 2a_{n-1}^2 + 1$ for $n > 0$. What is the last digit of a_{2022}?*

◯ *(A) 1*

◯ *(B) 2*

◯ *(C) 3*

◯ *(D) 5*

◯ *(E) 9*

Problem 9 *Given* 2022 *points in the plane no two of which share an* $x-$ *or* $y-$*coordinate, call one of them* m*-central if there are at least* m *other points to the left, to the right, above and below it. What is the largest* m *so that one is guaranteed to have an* m*-central point?*

◯ *(A) 0*

◯ *(B) 505*

◯ *(C) 506*

◯ *(D) 1010*

◯ *(E) 1011*

Problem 10 *Let* $\star$ *be a binary operation on the positive integers* $\mathbb{Z}^+$ *defined by* $1 \star n = n + 1$ *for* $n \geq 1$, $n \star 1 = (n-1) \star 2$ *for* $n > 1$, *and* $m \star n = (m-1) \star (m \star (n-1))$ *for* $m, n > 1$. *Determine* $3 \star 3$.

◯ *(A)* 1

◯ *(B)* 3

◯ *(C)* 5

◯ *(D)* 8

◯ *(E) None of the above*

Problem 11 *A cube with edge length* 2 *is painted black on the outside, then cut into* 8 *small cubes with edge length* 1 *which are placed in a box which is then shaken well. The light is then turned off, and the big cube is reassembled at random from the little cubes in the box. What is the probability that when the light is turned on again the big cube has an all-black surface?*

◯ *(A)* $\frac{1}{24^8}$

◯ *(B)* $\frac{1}{8^8}$

◯ *(C)* $\frac{8!}{24^8}$

◯ *(D)* $\frac{1}{8!}$

◯ *(E)* $\frac{1}{3^8}$

Problem 12 *The equation* $3 \cdot 2^m + 1 = n^2$ *has the following number of solutions in integer pairs* (m, n) *with* $m, n \geq 0$*:*

◯ *(A)* 1

◯ *(B)* 2

◯ *(C)* 3

◯ *(D)* 5

◯ *(E) Infinitely many*

Problem 13 *The sum*

$$\frac{2}{1 \cdot 2 \cdot 3} + \frac{3}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \ldots + \frac{2021}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2021 \cdot 2022}$$

is equal to

◯ *(A)* $\frac{1}{2} - \frac{1}{1 \cdot 2 \cdot \ldots \cdot 2022}$

◯ *(B)* $\pi/5$

◯ *(C)* $\frac{2}{3} + \frac{1}{1 \cdot 2 \cdot \ldots \cdot 2021}$

◯ *(D)* $\frac{5}{6} + \frac{1}{1 \cdot 2 \cdot \ldots \cdot 2021}$

◯ *(E)* 1

Problem 14 *Let* $n_1, n_2, \ldots$ *be positive integers whose sum is 100. What is the maximum possible value of the product of those integers?*

◯ *(A)* 2500

◯ *(B)* 2^{50}

◯ *(C)* 3^{33}

◯ *(D)* $2^{26} \cdot 3^{16}$

◯ *(E) None of the above*

Problem 15 *A fair six-sided die is rolled five times. What is the probability that each roll is at least as big as the previous one?*

◯ *(A)* $\frac{1}{120}$

◯ *(B)* $\frac{7}{432}$

◯ *(C)* $\frac{35}{1296}$

◯ *(D)* $\frac{1}{32}$

◯ *(E)* $\frac{7}{216}$

Problem 16 *Assume $a > b > c > d$ are positive integers such that*

$$a + b + c + d = 2022, \quad a^2 - b^2 + c^2 - d^2 = 2022.$$

What is the smallest value of a?

- ◯ *(A)* 230
- ◯ *(B)* 345
- ◯ *(C)* 507
- ◯ *(D)* 1011
- ◯ *(E) none of the above*

Problem 17 *Eight consecutive three-digit positive integers have the following property: each of them is divisible by its last digit. What is the product of the digits of the smallest of the eight integers?*

- ◯ *(A)* 10
- ◯ *(B)* 32
- ◯ *(C)* 56
- ◯ *(D)* 120
- ◯ *(E)* 240

Problem 18 *Three planes divide a unit sphere into 8 regions. The surface area of each region is computed and these are added together. What is the largest possible sum of the surface areas of these regions?*

- ◯ *(A)* 4π
- ◯ *(B)* 6π
- ◯ *(C)* 8π
- ◯ *(D)* 10π
- ◯ *(E)* 12π

Problem 19 *Consider the sequence* $x_0 = 1, x_1 = 2$ *and* $x_{n+2} = 2x_n x_{n+1} - x_n - x_{n+1} + 1$*. For how many values of* $0 \leq n \leq 1000$ *is* $2x_n - 1$ *a perfect square?*

◯ *(A)* 1

◯ *(B)* 30

◯ *(C)* 334

◯ *(D)* 667

◯ *(E) none of the above*

Problem 20 *Let* $A = 111\ldots1$ *be the number written with* 729 *ones. The remainder of* A *to division by* 729 *is*

◯ *(A)* 0

◯ *(B)* 9

◯ *(C)* 27

◯ *(D)* 726

◯ *(E)* 728

Problem 21 *Let* $t(n)$ *denote the largest power of* 2 *dividing* n*. What is* $\sum_{n=1}^{1023} t(n)$*?*

◯ *(A)* 5120

◯ *(B)* 5121

◯ *(C)* 5122

◯ *(D)* 5123

◯ *(E)* 5124

Problem 22 *Let* $f : \mathbb{Z} \to \mathbb{R}$ *be a function taking positive values such that for all* $n > 1$*, there exists a prime divisor* p *of* n *such that*

$$f(n)f(p) = f\left(\frac{n}{p}\right).$$

Assuming $f(210) = 3$*, what is the value of* $f(1001)$*?*

◯ *(A)* $\frac{1}{9}$

◯ *(B)* $\frac{1}{3}$

◯ *(C)* 1

◯ *(D)* $\sqrt{3}$

◯ *(E) none of the above*

Problem 23 *Let $ABCD$ be a parallelogram. A line through C, exterior to the parallelogram, intersects AB, AD in M, N. Three circles of radii r, r_1, r_2 are inscribed in the triangles AMN, NDC, MBC. Which of the following is necessarily true?*

◯ *(A)* $r = r_1 + r_2$

◯ *(B)* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$

◯ *(C)* $r^2 = r_1^2 + r_2^2$

◯ *(D)* $r = \frac{r_1^2+r_2^2}{r_1+r_2}$

◯ *(E) none of the above*

Problem 24 *Let n be a positive integer not all of whose base-10 digits are the same and none of which are 0. Suppose that no matter how n's digits are rearranged, it will always be divisible by k. What is the largest possible value of k?*

◯ *(A) 4*

◯ *(B) 36*

◯ *(C) 63*

◯ *(D) 72*

◯ *(E) 81*

Problem 25 *Two numbers are selected from 2 to 99. Serena is given the sum of the numbers, and Roger is given the product, without be told what the numbers are. Serena remarks to Roger: "I know that you don't know what the two numbers are". Roger replies "Now I know what the numbers are." Serena responds "Now I too know what the numbers are". What is the difference between the two numbers?*

◯ *(A)* 9

◯ *(B)* 12

◯ *(C)* 13

◯ *(D)* 25

◯ *(E)* 54