Name: ID Number:

Problem 1 *Matt has a biased coin that is more likely to come up heads than tails. He flips this coin n times and counts the number of tails. Show that this number is more likely to be even than it is to be odd.*

`Solution:` Let $p > 1/2$ be the probability that a single flip comes up heads. Let a_n be the probability that we have an even number of tails after n flips. Clearly $a_0 = 1$. For $n > 1$, we have an even number of tails if and only if we either had an even number after $n - 1$ flips and got a heads on the n^{th} flip, or had an odd number after $n - 1$ flips and got a tails. Thus,

$$a_n = pa_{n-1} + (1 - p)(1 - a_{n-1}) = (1 - p) + (2p - 1)a_{n-1}.$$

From here it is easy to see by induction on n that $a_n = 1/2 + (2p - 1)^n/2 > 1/2$.

Problem 2 *Let $n \geq 2$ be a positive integer and X be a set containing n^2 consecutive numbers. Let A be a subset of X with n elements. Show that $X \setminus A$ contains at least one arithmetic progression with n elements.*

`Solution`: Arrange the numbers in an $n \times n$ table in increasing order in each row and each column. If A "misses" a row or a column, we have an arithmetic progression; else, if A has exactly one element in each row/column, index with i_1 the column where the element of A is in row 1, with i_2 the column where the element of A is in row 2, etc. If any $i_{k+1} \geq i_k$, we have an arithmetic progression between the rows k and $k+1$. Else the only way to do it is by taking the anti-diagonal; in which case, the elements in the first upper-anti-diagonal (positions $(1,(n-1))$, $(2,(n-2))$, ... $((n-1),1)$) and $((n-1),n)$ give an arithmetic progression.

Problem 3 *Let $n > 0$ be an integer. It is known that the difference d of two divisors of 55^n is a power of 2. Show that $d = 4$.*

`Solution`: Let $1 \le d_1 < d_2$ be the two divisors whose difference is d. Write

$$d = d_2 - d_1 = 2^c.$$

The divisors d_1 and d_2 of 55^{50} can only contain the primes 5 and 11 in their factorization. Furthermore, d_1 and d_2 cannot be both divisible by 5 at the same time and they cannot be both divisible by 11 at the same time since their difference $d = 2^c$ is divisible neither by 5 nor by 11. Therefore,

$$d_1 = 5^a, d_2 = 11^b \text{ or } d_1 = 11^a, d_2 = 5^b.$$

We analyze the equations

$$5^a - 11^b = 2^c \text{ and } 11^a - 5^b = 2^c.$$

We show $c = 2$.

(i) Assume $11^a - 5^b = 2^c$ holds.

- Reducing the equation mod 5 we obtain

  $$2^c \equiv 1 \mod 5.$$

  Inspecting $c \mod 4$, we obtain $c \equiv 0 \mod 4$, so in particular

  $$c \text{ is even.}$$

- Noting that $11^a - 5^b$ is even we obtain that $c > 0$ so $c \ge 2$.
- Reducing the equation mod 4 we obtain

  $$(-1)^a - 1 \equiv 0 \mod 4 \implies a \text{ even.}$$

- Reducing the equation mod 3 we obtain

  $$(-1)^a - (-1)^b \equiv (-1)^c \mod 3.$$

  This cannot hold since a is even and c is even.

(ii) Assume $5^a - 11^b = 2^c$ holds.

- Reducing mod 3 we find

  $$(-1)^a - (-1)^b \equiv (-1)^c \mod 3$$

  which shows a, b cannot have the same parity.

- We reduce mod 8 to show that $c = 2$ is the only possibility.

  If a is even and b is odd, write

  $$a = 2k, \quad b = 2\ell + 1,$$

  and note that

  $$5^a - 11^b = 25^k - 11 \cdot 121^\ell \equiv 1 - 11 \cdot 1 \mod 8 \equiv 6 \mod 8 \implies 2^c \equiv 6 \mod 8.$$

  By inspection, this is impossible for $c \leq 2$. For $c \geq 3$ we obtain a contradiction since $2^c \equiv 0 \mod 8$.

  If a is odd and b is even, write

  $$a = 2k + 1, \quad b = 2\ell,$$

  and note that

  $$5^a - 11^b = 5 \cdot 25^k - 121^\ell \equiv 5 \cdot 1 - 1 \equiv 4 \mod 8 \implies 2^c \equiv 4 \mod 8.$$

  For $c \geq 3$ we obtain a contradiction since $2^c \equiv 0 \mod 8$. Thus $c \leq 2$. By inspection, $c = 2$ is the only possibility.

We showed $c = 2$ and therefore

$$d = 2^c = 4.$$

The difference $d = 4$ can be achieved for instance for the pairs $(1, 5)$ or $(121, 125)$.

Problem 4 *Let C be a set of n points on a circle in the plane. Prove that amongst any set of $n+1$ line segments between points in C, there exist two geometrically disjoint line segments, but that n segments are not sufficient.*

`Solution`: We prove the latter part first. In particular, we show that there is some set of n lines so that any pair intersect. We can do this for example by picking one of our points p and drawing the $n-1$ segments from p to each other point and the one segment between p's nearest neighbors on either side. The segments through p all intersect at p, and it is easy to see that the remaining segment intersects each of the others.

To show the other direction, for each of the n points p, select the segment (p, q) from our list where q is the closest point counting clockwise from p for which there is a segment. Since there are $n+1$ segments and only n points, some segment (p, q) must *not* have been selected. Since this was not the segment selected from p, there must be another segment q' so that we have a segment (p, q') in our set and p, q', q appear in that order going clockwise. Similarly, there must be a segment (p', q) so that q, p', p appear in that order going clockwise. Therefore, (p, q') and (p', q) are two segments in our set that do not intersect.