64th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 24-25, 2021

PART I
25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Calculators or other electronic devices may **not** be used. If you cannot conveniently work on a physical copy of the exam, you *may* use a computer but **only** to view the exam. You may not use an internet connection or any other programs on your computer.

2. You may use blank scratch paper, pens, pencils, compass and straight edge.

3. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.

4. Clearly indicate your answer in the bubble for each problem. Fill in the bubble, on the gradescope scantron sheet. You may write your answers on a separate sheet of paper and enter (but not change) them after time has ended.

Scoring
$+4$ points for a correct answer.
0 points for no answer.
-1 points for an incorrect answer.

Good Luck!

Please mark your calendars for the awards ceremony (held over zoom meeting) Saturday, May 1, 6:30–8:30pm.

Name: ID Number:

Problem 1 *Alice, Bob and Claire are exchanging marbles. First Alice gives half of her marbles to Bob. Then Bob gives half of his to Claire. Finally, Claire gives half of her marbles to Alice. After this is over, they realize that they each have the same number of marbles that they began with. What fraction of the marbles does Alice have?*

◯ *(A)* $1/4$

◯ *(B)* $1/3$

◯ *(C)* $2/5$

◯ *(D)* $1/2$

◯ *(E)* $2/3$

Problem 2 *Consider the sequences* $4, 8, 12, 16, \ldots$ *and* $2, 12, 22, 32, \ldots$*. Find the* 100^{th} *common term of the two sequences:*

◯ *(A)* 1992

◯ *(B)* 2012

◯ *(C)* 3992

◯ *(D)* 4012

◯ *(E) None of the above*

Problem 3 *Let* $ABCD$ *be an isosceles trapezoid with line* AB *parallel to* CD*, and segments* $AD = DC = CB = \frac{1}{2}AB$*. Let* E *be the midpoint of* AB*. Calculate the angle* $\angle EDB$*.*

◯ *(A)* $\pi/6$

◯ *(B)* $\pi/5$

◯ *(C)* $\pi/3$

◯ *(D)* $15°$

◯ *(E)* $45°$

Problem 4 *How many real-coefficient polynomials* $f(x)$ *satisfy* $f(x)^2 = f(f(x))$*?*

◯ *(A)* 0

◯ *(B)* 1

◯ *(C)* 2

◯ *(D) Infinitely Many*

◯ *(E) None of the above*

Problem 5 *Define a sequence by* $a_0 = 3$ *and* $a_n = a_{n-1}^2 + 1$ *for* $n > 0$*. What is the last digit of* a_{2021}*?*

◯ *(A)* 0

◯ *(B)* 3

◯ *(C)* 6

◯ *(D)* 7

◯ *(E)* 8

Problem 6 *Consider a* 2×100 *rectangle, and suppose we color each of its* 200 *unit squares with one of three colors (red, blue, or green). A coloring of the rectangle is called "proper" if all but one of the columns contain* 2 *differently colored squares. How many proper colorings are there?*

◯ *(A)* $100 \cdot 3^{101}$

◯ *(B)* $2^{101} \cdot 3^{100}$

◯ *(C)* $2^{104} \cdot 3^{102}$

◯ *(D)* $50 \cdot 6^{100}$

◯ *(E)* $300 \cdot 6^{100}$

Problem 7 *A convex pentagon has angles that are all an integer number of degrees and that also form an arithmetic progression. What is the smallest possible number of degrees that could be an angle of such a pentagon?*

◯ *(A)* 36

◯ *(B)* 37

◯ *(C)* 38

◯ *(D)* 39

◯ *(E)* 40

Problem 8 *An infinite cylinder with a radius of 1 is cut by a plane that makes a 45 degree angle with the center line of the cylinder. What is the area of the intersection?*

◯ *(A)* π

◯ *(B)* $\sqrt{2}\pi$

◯ *(C)* $3\pi/2$

◯ *(D)* $\sqrt{3}\pi$

◯ *(E)* 2π

Problem 9 *Consider an equilateral triangle of area 1. Cut it into 4 equal triangles, using lines parallel to the edges. Remove the center triangle. Continue the process, cutting each of the remaining 3 triangles in 4 equal equilateral triangles and removing the center one; repeat a total of 10 times (including the first). What is the total area removed?*

- ◯ *(A)* $\frac{3^{10}}{4^{10}}$
- ◯ *(B)* $\frac{3}{4}$
- ◯ *(C)* $1 - \frac{9^5}{16^5}$
- ◯ *(D)* $1 - \frac{3^{11}}{4^{11}}$
- ◯ *(E)* $1 - \frac{1}{16^5}$

Problem 10 *How many paths which don't self-intersect are there from* $(0, 0)$ *to* $(2, 2)$ *using only the steps* $(0, 1)$, $(0, -1)$, $(1, 0)$, $(-1, 0)$*?*

- ◯ *(A)* 6
- ◯ *(B)* 8
- ◯ *(C)* 10
- ◯ *(D)* 12
- ◯ *(E) None of the above*

Problem 11 *For integers* n *let* $s(n)$ *be the sum of the smaller factors of* n*. What is the smallest possible value of* $s(n)$ *among non-prime values of* $n > 2000$*?*

- ◯ *(A)* 1
- ◯ *(B)* 48
- ◯ *(C)* 89
- ◯ *(D)* 90
- ◯ *(E)* 91

Problem 12 *Let* $a_1 = 1, a_2 = 1$, *and* $a_{n+1} = a_n + a_{n-1}$. *For* $n \geq 1$, *let*

$$S_n = \frac{1}{a_1 \cdot a_3} + \frac{1}{a_2 \cdot a_4} + \ldots + \frac{1}{a_n \cdot a_{n+2}} .$$

The smallest n *such that* $S_n > \frac{99}{100}$ *is*

◯ *(A)* 5

◯ *(B)* 8

◯ *(C)* 9

◯ *(D)* 11

◯ *(E)* 13

Problem 13 *Let* $A = 5^{26} + 5^{27} + 5^{28} + \ldots + 5^{2021}$, *and let* $B = 31$. *What is the remainder when we divide* A *by* B?

◯ *(A)* 0

◯ *(B)* 25

◯ *(C)* 26

◯ *(D)* 28

◯ *(E)* 30

Problem 14 *What is the maximum number of bounded regions we can create by drawing 4 circles?*

◯ *(A)* 13

◯ *(B)* 14

◯ *(C)* 15

◯ *(D)* 16

◯ *(E) None of the above*

Problem 15 *We are given 10 coins, one of which is fake and of a different weight than the others (we don't know if it is lighter or heavier). We are also given a pair of balanced scales. To use the scales, we need to place coins on both sides. What is the smallest number of weighings we need to perform in order to* guarantee *we find the fake coin?*

◯ *(A)* 2

◯ *(B)* 3

◯ *(C)* 4

◯ *(D)* 5

◯ *(E)* 6

Problem 16 *Find the number of permutations* $(a_1, \ldots, a_{2021})$ *of the set* $\{1, 2, \ldots, 2021\}$ *with the property that there exists a unique index* $1 \leq i \leq 2020$ *such that* $a_i > a_{i+1}$.

◯ *(A)* $2^{2021} - 2022$

◯ *(B)* $2^{2021} - 2020$

◯ *(C)* $2020!$

◯ *(D)* $2^{1010} \cdot 2020!$

◯ *(E) None of the above*

Problem 17 *How many real solutions does the equation*

$$2^{x^2+x} + \log_2 x = 2^{x+1}$$

have?

◯ *(A)* 0

◯ *(B)* 1

◯ *(C)* 2

◯ *(D)* 5

◯ *(E) Infinitely many*

Problem 18 *Let a, b be integers. The line*

$$y = x \cdot \frac{a}{\sqrt{2}-1} + \frac{b}{\sqrt{2}+1}$$

passes through the point with coordinates $(\sqrt{2}+1, 5)$. Does the line pass though any points (x, y) with both coordinates x, y rational numbers?

◯ *(A) Yes, the point with coordinates* $(-2, -4)$

◯ *(B) Yes, the point with coordinates* $(2, 4)$

◯ *(C) Yes, the point with coordinates* $(-2, 4)$

◯ *(D) Yes, the point with coordinates* $(2, -4)$

◯ *(E) There are no such points on the line*

Problem 19 *Eight points are picked at random on the unit circle and line segments are drawn between every pair of these points. How many regions do these segments divide the plane into?*

◯ *(A)* 70

◯ *(B)* 72

◯ *(C)* 90

◯ *(D)* 92

◯ *(E)* 140

Problem 20 *Let $a = 4^{2022} + 1$. Solve for x:*

$$\sqrt{x + \sqrt{4x + \sqrt{4^2x + \sqrt{\ldots + \sqrt{4^{2021}x + a}}}}} - \sqrt{x} = 1.$$

◯ *(A)* 2^{2000}

◯ *(B)* 2^{2021}

◯ *(C)* 4^{2000}

◯ *(D)* 4^{2022}

◯ *(E) None of the above*

Problem 21 *A group of four people $\{a, b, c, d\}$ each possess a unique item of information. In some order, pairs of these people will call each other exchanging all the information that they know. What is the minimum number of calls needed in order for everyone to know every piece of information?*

◯ *(A)* 3

◯ *(B)* 4

◯ *(C)* 5

◯ *(D)* 6

◯ *(E)* 7

Problem 22 *Two points on Earth have longitudes that differ by 90 degrees and the central angle between the points is 120 degrees. What is the smallest possible difference between these points' latitudes?*

◯ *(A)* $30°$

◯ *(B)* $45°$

◯ *(C)* $60°$

◯ *(D)* $90°$

◯ *(E)* $\arccos(2/3)$

Problem 23 *For each integer $n \geq 2$, let $p(n)$ be the largest prime no larger than n, and $q(n)$ be the smallest prime bigger than n. The sum*

$$S = \frac{1}{p(2)q(2)} + \frac{1}{p(3)q(3)} + \ldots + \frac{1}{p(102)q(102)}$$

equals

◯ *(A)* $\frac{1}{2} - \frac{1}{101} - \frac{1}{2 \cdot 103}$

◯ *(B)* $\frac{1}{2} - \frac{1}{103}$

◯ *(C)* $\frac{1}{2} - \frac{1}{2 \cdot 101} + \frac{1}{103}$

◯ *(D)* $1 - \frac{1}{103}$

◯ *(E)* $1 - \frac{1}{101 \cdot 103}$

Problem 24 *What is the size of the largest subset* $S \subseteq \{1, 2, 3, \ldots, 100\}$ *so that there is no subset* $T \subseteq S$ *where the product of the elements of* T *is a perfect square?*

- ◯ *(A)* 10
- ◯ *(B)* 25
- ◯ *(C)* 26
- ◯ *(D)* 30
- ◯ *(E)* 90

Problem 25 *For how many irrational numbers* x*, are both* $x^3 + x$ *and* $x^6 + x^4 + 3x$ *integers?*

- ◯ *(A)* 0
- ◯ *(B)* 1
- ◯ *(C)* 2
- ◯ *(D)* 3
- ◯ *(E) Infinitely many*