63rd ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 25-26, 2020

PART II
4 Questions

Welcome to Part II of the contest!

Please do not open the exam until told do so by the proctor.

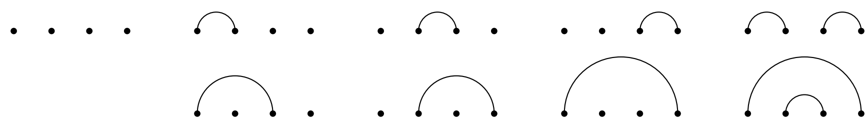
EXAMINATION DIRECTIONS:

1. Calculators or other electronic devices may **not** be used. If you cannot conveniently work on a physical copy of the exam, you *may* use a computer but **only** to view the exam and type your answers. You may not use an internet connection or any other programs on your computer.
2. You may use blank scratch paper, pens, pencils, compass and straight edge.
3. Part II consists of 4 problems, each worth 25 points. These problems are "essay" style questions. You should put all work towards a solution in the space following the problem statement. You should show all work and justify your responses as best you can.
4. Scoring is based on the progress you have made in understanding and solving the problem. The clarity and elegance of your response is an important part of the scoring.

Please mark your calendars for the awards ceremony (held over zoom meeting) Sunday, May 3, 3:00–5:30pm.

Name: ID Number:

Problem 1 *Draw n dots in a line. An* **arc diagram** *is a way to draw arcs (possibly none) that join some of the dots so that the arcs are all drawn above the line and dots and so that no two arcs intersect or share a dot. When $n = 4$, here are all of the arc diagrams:*



Let m_n be the number of ways an arc diagram can connect n dots (so by the above, $m_4 = 9$, and by convention, $m_0 = m_1 = 1$). Prove that for $n \geq 2$:

$$m_n = m_{n-1} + \sum_{i=0}^{n-2} m_i m_{n-2-i}.$$

Problem 2 *Find all real numbers x and positive integers n such that*

$$(1+(1+\sqrt{2})^x)^n+(1+(\sqrt{2}-1)^x)^n=8.$$

Name: ID Number:

Problem 3 *Let B denote the square $n \times n$ grid with each square colored red or white. Suppose that if a white square in B has at least two red neighboring squares (i.e. sharing a side with it), then that white square becomes red. This process is repeated until the coloring ceases changing. What is the minimum number of squares that would need to be colored red so that following this procedure would eventually lead the entire board to be red?*

Name: ID Number:

Problem 4 *Sage is playing tag with n of their friends. The playing area is an infinite plane Sage starts at the origin and their friends start at a point p two units away. Sage's friends each run at unit speed, while Sage can run at twice that. Sage's goal is run until they have caught each of their friends at least once. Show that the friends have a strategy which forces Sage to take at least $(1.01)^{\sqrt{n}-1}$ time in order to catch all of them.*