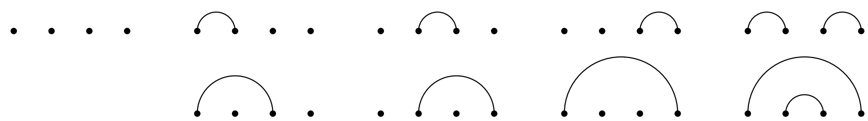
Name: ID Number:

Problem 1 *Draw n dots in a line. An* **arc diagram** *is a way to draw arcs (possibly none) that join some of the dots so that the arcs are all drawn above the line and dots and so that no two arcs intersect or share a dot. When $n = 4$, here are all of the arc diagrams:*



Let m_n be the number of ways an arc diagram can connect n dots (so by the above, $m_4 = 9$, and by convention, $m_0 = m_1 = 1$). Prove that for $n \geq 2$:

$$m_n = m_{n-1} + \sum_{i=0}^{n-2} m_i m_{n-2-i}.$$

Solution: Let $n \geq 2$. Each arc diagram on n dots is of one of two types: either the leftmost dot is in an arc or it isn't. For the first type, removing this dot gives a bijection with the set of arc diagrams on $n-1$ dots, so there are m_{n-1} arc diagrams of this form.

For the second type, we can further split them based on which other dot the first dot is connected to. Let i be the number of dots underneath this arc. Then the number of diagrams with this value of i is $m_i m_{n-2-i}$ because no arcs can go between these i dots and the dots to the right of the first arc because of the non-intersecting condition, and so we are picking an arc diagram just on these i dots together with an arc diagram on the last $n-2-i$ dots. This value of i can be anything from 0 to $n-2$, so the second type of diagrams is counted by $\sum_{i=0}^{n-2} m_i m_{n-2-i}$.

Problem 2 *Find all real numbers x and positive integers n such that*

$$(1+(1+\sqrt{2})^x)^n + (1+(\sqrt{2}-1)^x)^n = 8.$$

Solution: The solutions are $x=0, n=2$ and $x=\pm 2, n=1$.

Indeed, let $a = 1+\sqrt{2}$ so that $a^{-1} = \sqrt{2}-1$. The equation to be solved is

$$(1+a^x)^n + (1+a^{-x})^n = 8. \tag{1}$$

We have

$$(a^{x/2}-1)^2 \geq 0 \implies 1+a^x \geq 2\cdot a^{x/2} \implies (1+a^x)^n \geq 2^n \cdot a^{nx/2} \tag{2}$$

and similarly

$$(1+a^{-x})^n \geq 2^n \cdot a^{-nx/2}. \tag{3}$$

Next, we have

$$(a^{nx/4} - a^{-nx/4})^2 \geq 0 \implies a^{nx/2} + a^{-nx/2} \geq 2.$$

Thus

$$8 = (1+a^x)^n + (1+a^{-x})^n \geq 2^n \cdot (a^{nx/2} + a^{-nx/2}) \geq 2^{n+1}.$$

This shows that $n=1$ or $n=2$.

If $n=2$, we must have equality throughout. In particular, from (2), we obtain

$$a^x = 1 \implies x = 0.$$

If $n=1$, equation (1) becomes

$$a^x + a^{-x} = 6.$$

Set $y = a^x$. Then

$$y + \frac{1}{y} = 6 \iff y^2 - 6y + 1 = 0 \iff y = 3 \pm 2\sqrt{2} \iff a^x = 3 \pm 2\sqrt{2}.$$

We can directly verify that

$$(1+\sqrt{2})^2 = 3+2\sqrt{2} \text{ and } (1+\sqrt{2})^{-2} = (3+2\sqrt{2})^{-1} = 3-2\sqrt{2}.$$

This implies that $x = \pm 2$.

Problem 3 *Let B denote the square $n \times n$ grid with each square colored red or white. Suppose that if a white square in B has at least two red neighboring squares (i.e. sharing a side with it), then that white square becomes red. What is the minimum number of squares that would need to be colored red so that following this procedure would eventually lead the entire board to be red?*

Solution: The answer is n. It is easy to see that you can color n red squares to convert the entire grid, by coloring the squares along a long diagonal red. After one round, the squares on the diagonals immediately above and below will be red, and the next round the diagonals below that and so on.

To show that you cannot achieve this with fewer red squares, we note that every time a white square adjacent to two red ones is colored red, the perimeter of the region defined by the set of red squares does not get any bigger. This is because at most two new edges are added to this perimeter, while at least two edges are removed. Thus, if you want to end up with the entire grid (which has perimeter $4n$) being red, your initial collection of s squares must have perimeter at least $4n$. However, such a set cannot have perimeter more than $4s$, so we get that s must be at least n.

Problem 4 *Sage is playing tag with n of their friends. The playing area is an infinite plane Sage starts at the origin and their friends start at a point p two units away. Sage's friends each run at unit speed, while Sage can run at twice that. Sage's goal is run until they have caught each of their friends at least once. Show that the friends have a strategy which forces Sage to take at least $(1.01)^{\sqrt{n}-1}$ time in order to catch all of them.*

Solution: The strategy is as follows. Let $m = \lceil \sqrt{n} \rceil$. Each of Sage's friends picks a distinct pair of integers (a, b) with $1 \leq a, b \leq m$ (this is possible since $m^2 \geq n$). Assume that Sage starts at point $(-2, 0)$ in the plane and the friends start at point $p = (0, 0)$. Then the friend with numbers (a, b) can run so that at time t they are in location $(at/m\sqrt{2}, bt/m\sqrt{2})$. We note that this involves them running at speed $\sqrt{\frac{a^2+b^2}{2m^2}} \leq 1$ and so is a possible strategy.

When responding to this, suppose that Sage catches their i^{th} friend at time t_i. Since all the friends are running to locations with positive x-coordinate, it will take Sage at least unit time to reach any, so $t_1 \geq 1$. If Sage catches their i^{th} friend at time t_i, then at that time all other friends must be at distance at least $t_i/(m\sqrt{2})$ from Sage. Since the sum of Sage's speed with that of any friends is at most 3, it must take Sage an additional time of at least $t_i/(3m\sqrt{2})$ to catch their next friend. Thus, $t_{i+1} \geq t_i(1 + 1/(3m\sqrt{2}))$. Therefore, we conclude that no matter what strategy Sage uses, it will take time at least $t_n \geq (1 + 1/(3m\sqrt{2}))^{n-1}$ to catch all of their friends.

We note that

$$(1 + 1/(3m\sqrt{2}))^m = \sum_{k=0}^{m} \binom{m}{k} (1/(3m\sqrt{2}))^k \geq 1 + m/(3m\sqrt{2}) = 1 + 1/(3\sqrt{2}) > 1.1.$$

Therefore, we have that the time Sage requires is at least $(1.1)^{(n-1)/m}$. Since $m \leq \sqrt{n} + 1$, we have that $(n-1)/\sqrt{m} \geq \sqrt{n} - 1$, and so we have that the total time required is at least $(1.1)^{\sqrt{n}-1}$.