63rd ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 25-26, 2020

PART I
25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Calculators or other electronic devices may **not** be used. If you cannot conveniently work on a physical copy of the exam, you *may* use a computer but **only** to view the exam. You may not use an internet connection or any other programs on your computer.

2. You may use blank scratch paper, pens, pencils, compass and straight edge.

3. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.

4. Clearly indicate your answer in the bubble for each problem. Fill in the bubble, on the gradescope scantron sheet. You may write your answers on a separate sheet of paper and enter (but not change) them after time has ended.

Scoring
$+4$ points for a correct answer.
0 points for no answer.
-1 points for an incorrect answer.

Good Luck!

Please mark your calendars for the awards ceremony (held over zoom meeting) Sunday, May 3, 3:00–5:30pm.

Name: ID Number:

Problem 1 *Given n students, some enroll in history and some enroll in math. Every student must take at least 1 class. How many different ways can this happen?*

◯ *(A)* $\binom{n}{2}$

◯ *(B)* $\binom{n}{3}$

◯ *(C)* 2^n

◯ *(D)* 3^n

◯ *(E)* 4^n

Problem 2 *Two dice are rolled. What is the probability that their values are consecutive numbers?*

◯ *(A)* $\frac{1}{2}$

◯ *(B)* $\frac{1}{3}$

◯ *(C)* $\frac{5}{9}$

◯ *(D)* $\frac{5}{18}$

◯ *(E)* $\frac{5}{36}$

Problem 3 *Pat has 8 gallons of 1% milk and 4 gallons of $3\,^1\!/_2\%$ milk. What is the maximum number of gallons of 2% milk could they produce by mixing them?*

◯ *(A)* 10

◯ *(B)* 11

◯ *(C)* $\frac{40}{3}$

◯ *(D)* 12

◯ *(E)* $\frac{589}{49}$

Problem 4 *Let $a_0 = 24$. A sequence is constructed recursively by letting a_{n+1} be given by first adding 1 to the sum of digits of a_n and then squaring. For instance,*

$$a_1 = (2+4+1)^2 = 49, \quad a_2 = (4+9+1)^2 = 196.$$

What is the value of a_{2020}?

◯ *(A)* 25

◯ *(B)* 49

◯ *(C)* 121

◯ *(D)* 196

◯ *(E)* 400

Problem 5 *For an integer n, let $f(n)$ denote the product of the digits of n when written in base 10 with no leading zeroes. So for example $f(46) = 4 \cdot 6 = 24$. What is the sum $f(1) + f(2) + \ldots + f(100)$?*

◯ *(A)* 2025

◯ *(B)* 2026

◯ *(C)* 2070

◯ *(D)* 2115

◯ *(E)* 2116

Problem 6 *For which values of a does the equation $a|x| = 4 - 2x$ have exactly 2 solutions?*

◯ *(A) Always*

◯ *(B)* $(-\infty, -2)$

◯ *(C)* $(-2, 2)$

◯ *(D)* $(-2, 0) \cup (0, 2)$

◯ *(E)* $(2, \infty)$

Problem 7 *How many 4 digit numbers in base 10 have exactly one digit equal to 0 when written in base 2?*

◯ *(A)* 21

◯ *(B)* 37

◯ *(C)* 38

◯ *(D)* 41

◯ *(E)* 42

Problem 8 *Find the number of elements of the largest set $A \subset \{1, 2, \ldots, 2020\}$ with the property that for all $a, b \in A$, the difference $a - b$ does not divide the sum $a + b$.*

◯ *(A)* 673

◯ *(B)* 674

◯ *(C)* 1009

◯ *(D)* 1010

◯ *(E)* 1505

Problem 9 *How many integers between* 1 *and* 1000 *are neither perfect squares nor perfect cubes?*

◯ *(A)* 952

◯ *(B)* 954

◯ *(C)* 959

◯ *(D)* 962

◯ *(E)* 966

Problem 10 *How many pairs of integers* (x, y) *with* $1 \leq x < y \leq 2020$ *have the property that* x, y *have exactly* $y - x$ *common (positive) divisors?*

◯ *(A)* 1009

◯ *(B)* 2019

◯ *(C)* 2020

◯ *(D)* 3028

◯ *(E)* 3029

Problem 11 *Let* $c > 0$*. In a coordinate system, consider the following three lines*

$$\ell_1 : \quad y = -\frac{x}{2} + 1,$$

$$\ell_2 : \quad y = -\frac{x}{5} + 1,$$

$$\ell_3 : \quad y = -\frac{x}{c} + 1.$$

The angles α, β, γ *between these lines and the* x*-axis satisfy*

$$\alpha + \beta + \gamma = \frac{\pi}{4}.$$

The value of c *is*

◯ *(A)* 1

◯ *(B)* 2

◯ *(C)* 3

◯ *(D)* 8

◯ *(E) There is not enough information to determine*

Problem 12 *How many four digit palindromes are divisible by 7?*

◯ *(A)* 9

◯ *(B)* 12

◯ *(C)* 13

◯ *(D)* 14

◯ *(E)* 18

Problem 13 *Let L be a line tangent to a circle C of area 1. Let C_1 be another circle with area 1 so that C_1 is externally tangent to both C and L. Define a sequence of circles $C_2, C_3, \ldots$ so that C_i is externally tangent to L, C and C_{i-1}. What is the area of C_{2020}?*

◯ *(A)* 1

◯ *(B)* $1/2020$

◯ *(C)* $1/2020^2$

◯ *(D)* $1/2020^3$

◯ *(E)* $1/2020^4$

Problem 14 *Jesse is running on a track. Half of the other runners on the track are going the same direction as Jesse and half in the other direction. All other runners are going at the same speed. Jesse realizes that they are passing runners travelling in the opposite direction twice as often as they are passing runners traveling in the same direction. How many times faster is Jesse travelling than the other runners?*

◯ *(A)* 1

◯ *(B)* 2

◯ *(C)* 3

◯ *(D)* 4

◯ *(E)* 5

Problem 15 *What is the largest integer k such that if R is any room with k walls, then there exists a place to stand in the room from which all parts of the room are visible?*

◯ *(A)* 3

◯ *(B)* 4

◯ *(C)* 5

◯ *(D)* 6

◯ *(E)* 7

Problem 16 *For real numbers x, y, z, w each of absolute value at least 1, what is the maximum number of possible subsets of $\{x, y, z, w\}$ whose sum is between 0 and 1 inclusive? For example if $(x, y, z, w) = (-2, 1.1, 1.3, 2.5)$, then the sum of none of them is 0, $x+w = 0.5$, and $x+y+z = 0.4$ so there would be three such subsets (the empty set, $\{x, w\}$ and $\{x, y, z\}$).*

◯ *(A)* 3

◯ *(B)* 4

◯ *(C)* 5

◯ *(D)* 6

◯ *(E)* 7

Problem 17 *The diagonals of the parallelogram $ABCD$ intersect in O. The angle bisectors of $\angle DAO$ and $\angle OBC$ intersect in M. If $DOCM$ is a parallelogram then $\frac{AB}{AD}$ equals*

◯ *(A)* $\frac{1}{\sqrt{3}}$

◯ *(B)* $\frac{2}{\sqrt{5}}$

◯ *(C)* 1

◯ *(D)* $\frac{\sqrt{5}}{2}$

◯ *(E)* $\sqrt{3}$

Problem 18 *Let A, B, C be three clubs of three people, such that $A \cap B = A \cap C = B \cap C = \{p\}$. If the people in $A \cup B \cup C$ are randomly assigned distinct numbers, then what is the probability that p is not the largest number in any of the clubs A, B, C?*

◯ *(A)* 0

◯ *(B)* $\frac{1}{8}$

◯ *(C)* $\frac{16}{35}$

◯ *(D)* $\frac{13}{30}$

◯ *(E)* $\frac{3}{5}$

Problem 19 *What is the value of $\sqrt[3]{9 + 4\sqrt{5}} + \sqrt[3]{9 - 4\sqrt{5}}$?*

◯ *(A)* 2

◯ *(B)* $2\sqrt{2}$

◯ *(C)* 3

◯ *(D)* 4

◯ *(E) None of the above*

Problem 20 *Consider the square $ABCD$ and M, N two interior points such that*

$$AM = a,\ MN = b,\ NC = c \text{ and } AM \perp MN,\ MN \perp CN.$$

The area of the square equals

◯ *(A)* $\frac{b^2+(a+c)^2}{2}$

◯ *(B)* $\frac{c^2(b^2+(a+c)^2)}{(a+c)^2}$

◯ *(C)* $a^2 + b^2 + c^2$

◯ *(D)* $ab + bc$

◯ *(E) none of the above*

Problem 21 *The cards of a standard deck of 52 cards are dealt one by one. Before a card is dealt, you must guess "red", "black", or "pass". If the color of the card is guessed correctly, you gain one point. If the color is guessed incorrectly, you lose 1 point. If you passed, then you get 0 points. If the dealer is allowed to pick the card order in order to allow you as few points as possible, what is the maximum number of points that you can guarantee?*

◯ *(A)* 0

◯ *(B)* 1

◯ *(C)* 2

◯ *(D)* 5

◯ *(E)* 13

Problem 22 *Define a sequence by $a_0 = 1/2020$, and $a_{n+1} = a_n(a_n + 1)$ for all $n \geq 0$. What is the smallest value of n so that $a_n \geq 1$?*

◯ *(A)* 10

◯ *(B)* 2020

◯ *(C)* 2027

◯ *(D)* 4040

◯ *(E) There is no such n*

Problem 23 *Let n be a positive integer with exactly 505 positive divisors $d_1, \ldots, d_{505}$. Find the maximum possible value of the sum*

$$S = \frac{1}{d_1^2 + n} + \ldots + \frac{1}{d_{505}^2 + n}.$$

◯ *(A)* $\frac{253}{2^5 . 3^{100}}$

◯ *(B)* $\frac{505}{2^{101} . 3^4}$

◯ *(C)* $\frac{505}{2^{505}}$

◯ *(D)* $\frac{505}{2^4 . 3^{100}}$

◯ *(E) there is no maximum value*

Problem 24 *Which of following shapes can be drawn in the plane with integer co-ordinates?*

◯ *(A) A regular triangle*

◯ *(B) A regular pentagon*

◯ *(C) A regular hexagon*

◯ *(D) A regular octagon*

◯ *(E) None of the above*

Problem 25 *What is $1 + 1$?*

◯ *(A)* $\max_{n=1,2,\ldots,1000000}(\sin(n) + \cos(n))^2$ *(the arguments here are in radians)*

◯ *(B) The first digit of* 2^{280}

◯ *(C) The number of primes all of whose decimal digits are* 2

◯ *(D) The remainder when* 2020^{20} *is divided by* 7

◯ *(E) The smallest positive integer not divisible by positive integers other than itself and* 1