62nd ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 20, 2019
on the campus of the
University of California, San Diego

PART II
4 Questions

Welcome to Part II of the contest!

Please print your Name, School, and Contest ID number:

Name _____

First   Last

School _____

3-digit Contest ID number   _____

Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number on **each page of the contest**.

2. Part II consists of 4 problems, each worth 25 points. These problems are "essay" style questions. You should put all work towards a solution in the space following the problem statement. You should show all work and justify your responses as best you can.

3. Scoring is based on the progress you have made in understanding and solving the problem. The clarity and elegance of your response is an important part of the scoring. You may use the back side of each sheet to continue your solution, but be sure to call the reader's attention to the back side if you use it.

4. Give this entire exam to a proctor when you have completed the test to your satisfaction.

Please let your coach know if you plan to attend the Awards Banquet on Sunday, April 28, 6:00–8:30pm at the UCSD Price Center.

Problem 1 *Let* $\mathbb{N} = \{1, 2, 3, \ldots\}$ *denote the set of positive integers. Let* $f : \mathbb{N} \to \mathbb{N}$ *be a strictly increasing function so that* $f(2) = 2$ *and* $f(xy) = f(x)f(y)$ *for all* $x, y \in \mathbb{N}$. *Show that* $f(n) = n$ *for all* $n \in \mathbb{N}$.

Problem 2 *Show that all positive integers except those which are powers of two can be represented as a sum of at least two consecutive positive integers.*

Problem 3 *Given $2n$ cards, we shuffle them by taking the first n cards and interlacing them with the second n cards so that the top card stays on top. For example, if we had 8 cards, then the new order would be $1, 5, 2, 6, 3, 7, 4, 8$. If we have 1002 cards, then what is the least number of shuffles needed before the cards are back in their original order? (Zero is not the answer we are looking for- assume that the cards are shuffled at least once.)*

Problem 4 *Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be distinct sets of positive integers. Assume that any integer m can be written as $a_i + a_j$ with $1 \le i < j \le n$ in exactly as many ways as it can be written as $b_i + b_j$ with $1 \le i < j \le n$. Show that n is a power of 2.*