62nd ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 20, 2019
on the campus of the
University of California, San Diego

PART II
4 Questions

Welcome to Part II of the contest!

Please print your Name, School, and Contest ID number:

Name ____
First   Last

School ____

3-digit Contest ID number ____________________________

Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number on **each page of the contest**.
2. Part II consists of 4 problems, each worth 25 points. These problems are "essay" style questions. You should put all work towards a solution in the space following the problem statement. You should show all work and justify your responses as best you can.
3. Scoring is based on the progress you have made in understanding and solving the problem. The clarity and elegance of your response is an important part of the scoring. You may use the back side of each sheet to continue your solution, but be sure to call the reader's attention to the back side if you use it.
4. Give this entire exam to a proctor when you have completed the test to your satisfaction.

Please let your coach know if you plan to attend the Awards Banquet on Sunday, April 28, 6:00–8:30pm at the UCSD Price Center.

Name: ID Number:

Problem 1 *Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function so that $f(2) = 2$ and $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{N}$. Show that $f(n) = n$ for all $n \in \mathbb{N}$.*

Solution: We prove by induction on n that for all $m \leq 2n$ that $f(m) = m$. For the base case $n = 1$, we note that $f(2) = 2$. Since f is strictly increasing $f(1)$ must be 1. Next suppose that $f(m) = m$ for all $m \leq 2n$. Then note that $f(2(n+1)) = f(2)f(n+1) = 2f(n+1) = 2(n+1)$. Furthermore we must have that $2n = f(2n) < f(2n+1) < f(2n+2) = 2n+2$, so $f(2n+1)$ must equal $2n+1$. This proves that $f(m) = m$ for all $m \leq 2(n+1)$.

This completes the inductive step and proves our result.

Problem 2 *Show that all positive integers except those which are powers of two can be represented as a sum of at least two consecutive positive integers.*

Solution: First note that a sum of consecutive integers of the form $a + (a + 1) + \ldots + b$ is equal to $(a + b)(b - a + 1)/2$.

Next, for n not a power of 2, we may write $2n = xy$ where x is a power of 2 and y is odd and both x and y are bigger than 1. In particular, we may write $2n = xy$ where one of x and y is even and the other is odd and both are bigger than 1. Suppose without loss of generality that $x > y$. Then we may let $a = (x - y + 1)/2$ and $b = (x + y - 1)/2$. Note that since $x + y$ and $x - y$ are odd that both a and b are integers. Since $y > 1$, we have that $b > a$. Finally, $a + (a+1) + \ldots + b = (a+b)(b-a+1)/2 = [(x-y+1)/2 + (x+y-1)/2][(x+y-1)/2 - (x-y+1)/2 + 1]/2 = xy/2 = 2n/2 = n$.

This completes the proof.

Problem 3 *Given $2n$ cards, we shuffle them by taking the first n cards and interlacing them with the second n cards so that the top card stays on top. For example, if we had 8 cards, then the new order would be $1, 5, 2, 6, 3, 7, 4, 8$. If we have 1002 cards, then what is the least number of shuffles needed before the cards are back in their original order? (Zero is not the answer we are looking for-assume that the cards are shuffled at least once.)*

Solution: Number the cards 0 through $2n-1$. Ignoring the very last card, which is always fixed in place, our shuffle sends card i to position $2i$ taken modulo $2n-1$. So our question is equivalent to: what is the smallest positive d such that $2^d \equiv 1 \pmod{1001}$? First, the prime factorization of 1001 is $7 \cdot 11 \cdot 13$. By the Chinese remainder theorem, $\mathbf{Z}/1001$ is isomorphic to $\mathbf{Z}/7 \times \mathbf{Z}/11 \times \mathbf{Z}/13$. We solve the problem modulo $7, 11, 13$:

- The powers of 2 in $\mathbf{Z}/7$ are $2, 4, 1$, so $2^3 \equiv 1 \pmod 7$ and that is the smallest power equal to 1.
- The powers of 2 in $\mathbf{Z}/11$ are $2, 4, 8, 5, 10, 9, 7, 3, 6, 1$, so $2^{10} \equiv 1 \pmod{11}$ is the smallest power equal to 1.
- The powers of 2 in $\mathbf{Z}/13$ are $2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1$, so $2^{12} \equiv 1 \pmod{13}$ is the smallest power equal to 1.

Our desired answer is $\operatorname{lcm}(3, 10, 12) = 60$.

Problem 4 *Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be distinct sets of positive integers. Assume that any integer m can be written as $a_i + a_j$ with $1 \le i < j \le n$ in exactly as many ways as it can be written as $b_i + b_j$ with $1 \le i < j \le n$. Show that n is a power of 2.*

Solution: Set
$$f(x) = \sum_{i=1}^{n} x^{a_i} \quad \text{and} \quad g(x) = \sum_{i=1}^{n} x^{b_i}.$$
We compute
$$\begin{aligned} f(x)^2 - f(x^2) &= \left(\sum_{i=1}^{n} x^{a_i}\right)^2 - \left(\sum_{i=1}^{n} x^{2a_i}\right) = 2 \sum_{1\le i<j\le n} x^{a_i+a_j}, \\ g(x)^2 - g(x^2) &= \left(\sum_{i=1}^{n} x^{b_i}\right)^2 - \left(\sum_{i=1}^{n} x^{2b_i}\right) = 2 \sum_{1\le i<j\le n} x^{b_i+b_j}. \end{aligned}$$
Therefore,
$$f(x)^2 - f(x^2) = g(x)^2 - g(x^2).$$
Indeed, by by the assumptions of the problem, the coefficient of x^m is the same on both sides, for all m.

Note furthermore that
$$f(1) = g(1) = n \implies f(1) - g(1) = 0.$$
The polynomial $f(x) - g(x)$ is divisible by $x - 1$, and factoring out the largest power of $x - 1$ we can write
$$f(x) - g(x) = (x-1)^k p(x), \quad k \ge 1, \;\; p(1) \ne 0, \;\; p \in \mathbb{Z}[x].$$
Substituting we obtain
$$\begin{aligned} f(x)^2 - g(x)^2 &= f(x^2) - g(x^2) \implies \\ (f(x) - g(x))(f(x) + g(x)) &= f(x^2) - g(x^2) \implies \\ (x-1)^k p(x)\,(f(x) + g(x)) &= (x^2-1)^k p(x^2) \implies \\ p(x)\,(f(x) + g(x)) &= (x+1)^k p(x^2). \end{aligned}$$
Setting $x = 1$ we find
$$p(1)\cdot(f(1) + g(1)) = 2^k \cdot p(1) \implies f(1) + g(1) = 2^k \implies 2n = 2^k \implies n = 2^{k-1}.$$
Thus n is a power of 2.