62nd ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 20, 2019
on the campus of the
University of California, San Diego

PART I
25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number at the top of **each page of the contest**. (Keep your ID number handy for Part II.)
2. Calculators or other electronic devices may **not** be used.
3. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.
4. Clearly indicate your answer in the bubble for each problem. Fill in the bubble, or put a clear X, or a checkmark; just make sure your selection is clear.
5. At the end of Part I, tear off this front page, and turn in your (still-stapled) contest.

Scoring
$+4$ points for a correct answer.
0 points for no answer.
-1 points for an incorrect answer.

Good Luck!

There will be a 15 minute break after Part I before proceeding to Part II.

Please let your coach know if you plan to attend the Awards Banquet on Sunday, April 28, 6:00–8:30pm at the UCSD Price Center.

Name: ID Number:

Problem 1 *A hexagon has one angle of* $90°$ *and five other equal angles. How many degrees are the other angles?*

- ◯ *(A) 90*
- ◯ *(B) 120*
- ◯ *(C) 126*
- ◯ *(D) 132*
- ◯ *(E) 145*

Problem 2 *What is the solution in* x *to* $10^x \cdot 1000^x = 100000000$*?*

- ◯ *(A) 1*
- ◯ *(B)* $\frac{3}{2}$
- ◯ *(C) 2*
- ◯ *(D)* $\frac{5}{2}$
- ◯ *(E) 3*

Problem 3 *Let* $ABCD$ *be a parallelogram with* E *and* F *the midpoints of* AB *and* CD*, respectively. What is the ratio of the area of* $ABCD$ *to that of* $EBFD$*?*

- ◯ *(A) 1*
- ◯ *(B)* $\sqrt{2}$
- ◯ *(C)* $\frac{3}{2}$
- ◯ *(D)* 2
- ◯ *(E) 3*

Problem 4 *How many two-digit numbers have distinct and nonzero digits?*

- ◯ *(A) 72*
- ◯ *(B) 80*
- ◯ *(C) 81*
- ◯ *(D) 90*
- ◯ *(E) 100*

Problem 5 *How many real solutions does the equation* $\sqrt{x+5} - \sqrt{x-2} = -1$ *have?*

◯ *(A) 0*

◯ *(B) 1*

◯ *(C) 2*

◯ *(D) 3*

◯ *(E) 4*

Problem 6 *How many permutations* (x_1, x_2, x_3, x_4) *of the set* $\{1, 2, 3, 4\}$ *have the property that*

$$x_1x_2 + x_1x_3 + x_1x_4$$

is divisible by 3?

◯ *(A) 8*

◯ *(B) 12*

◯ *(C) 16*

◯ *(D) 18*

◯ *(E) 24*

Problem 7 *Consider the sequence of digits 1234567891011121314151617181920...where the integers in order are written in base 10 in order and their digits placed one after the other. What is the 2019th digit in this sequence?*

◯ *(A) 0*

◯ *(B) 2*

◯ *(C) 5*

◯ *(D) 7*

◯ *(E) 9*

Problem 8 *Let* n *be a positive integer. What is the number of functions* $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ *which satisfy* $f(k) \geq k$ *for every* $k \in \{1, 2, \ldots, n\}$*?*

◯ *(A)* $\frac{2n^2-7n+7}{2}$

◯ *(B)* $\frac{4^n+2}{3}$

◯ *(C)* $n^{n/2}$

◯ *(D)* $n!$

◯ *(E)* n^n

Problem 9 *A fair six sided die is rolled* 3 *times. What is the probability that the product of the rolls is a power of* 2*?*

- ◯ *(A)* 0
- ◯ *(B)* $\frac{5}{216}$
- ◯ *(C)* $\frac{1}{27}$
- ◯ *(D)* $\frac{1}{8}$
- ◯ *(E)* $\frac{3}{2}$

Problem 10 *How many integer solutions does* $ab - 3b - 2a = 7$ *have?*

- ◯ *(A) 0*
- ◯ *(B) 1*
- ◯ *(C) 2*
- ◯ *(D) 4*
- ◯ *(E) Infinitely many*

Problem 11 *What is the number of ordered pairs* (A, B) *of subsets of* $\{1, 2, \ldots, n\}$ *with nonempty intersection?*

- ◯ *(A)* $2^n - 1$
- ◯ *(B)* $3^n - 1$
- ◯ *(C)* $3^n - 2^n$
- ◯ *(D)* $4^n - 2^n$
- ◯ *(E)* $4^n - 3^n$

Problem 12 *A square* S *of side length* 1 *is rotated about its center. In doing so, it traces out a figure* F*. What is the area of* F*?*

- ◯ *(A) 1*
- ◯ *(B) 2*
- ◯ *(C)* $\frac{\pi}{2}$
- ◯ *(D)* π
- ◯ *(E)* 2π

Problem 13 *How many divisors of the number $A = 20!$ are perfect squares?*

◯ *(A) 72*

◯ *(B) 144*

◯ *(C) 150*

◯ *(D) 300*

◯ *(E) 600*

Problem 14 *Find the remainder attained by dividing the polynomial*

$$x^{2019} - 2x^{1992} + x^{1965} + x^{55} + 2x^{28} + x^5 + 2019$$

by $(x^{18} + x^9 + 1)(x^6 + x^3 + 1)$.

◯ *(A) 0*

◯ *(B) 2019*

◯ *(C)* $x^5 + 2x + 2018$

◯ *(D)* $x^5 + 3x + 2019$

◯ *(E)* $x^{55} + x^5 + 2018$

Problem 15 *Let P be a pentagon obtained by starting with a square and attaching an equilateral triangle along the top edge. What is the area of P divided by the square of its perimeter?*

◯ *(A)* $(\sqrt{3} - 1)/50$

◯ *(B)* $(4 + \sqrt{3})/100$

◯ *(C)* $(1 + \sqrt{5})/54$

◯ *(D)* $1/10$

◯ *(E)* $(1 + \sqrt{3})/25$

Problem 16 *Let a and b be the roots of the polynomial $x^2 - x - 5$. What is $a^4 + b^4$?*

◯ *(A) 16*

◯ *(B) 66*

◯ *(C)* $6 + 12\sqrt{21}$

◯ *(D) 71*

◯ *(E) 81*

Problem 17 *What is the median number on this list?*

- ◯ *(A)* 20^{19}
- ◯ *(B)* 2^{90^1}
- ◯ *(C)* 10^{29}
- ◯ *(D)* 201^9
- ◯ *(E)* 19^{20}

Problem 18 *How many 4 digit numbers have some pair of consecutive digits the same?*

- ◯ *(A) 2400*
- ◯ *(B) 2430*
- ◯ *(C) 2439*
- ◯ *(D) 2700*
- ◯ *(E) 3000*

Problem 19 *In the xy-plane what is the length of the shortest path from $(0, 0)$ to $(12, 16)$ that does not enter the interior of the circle defined by $(x - 6)^2 + (y - 8)^2 = 25$?*

- ◯ *(A)* $5\sqrt{3} + \frac{5\pi}{3}$
- ◯ *(B) 20*
- ◯ *(C)* $10\sqrt{3} + \frac{5\pi}{3}$
- ◯ *(D)* $\frac{40\sqrt{3}}{3}$
- ◯ *(E)* $10 + 5\pi$

Problem 20 *Consider the set S of positive integers between 1 and 1,000,000 inclusive. Let A be the subset of S consisting of numbers which can be written in the form $a^2 + b^4$ for positive integers a and b. What can you say about the size of A?*

- ◯ *(A)* $|A| < 30,600$
- ◯ *(B)* $30,600 < |A| < 32,000$
- ◯ *(C)* $32,000 < |A| < 100,000$
- ◯ *(D)* $100,000 < |A| < 500,000$
- ◯ *(E)* $500,000 < |A|$

Problem 21 *Let*

$$S_n = \frac{1}{\cos(0)\cos(\frac{\pi}{2019})} + \frac{1}{\cos(\frac{\pi}{2019})\cos(\frac{2\pi}{2019})} + \cdots + \frac{1}{\cos(\frac{(n-1)\pi}{2019})\cos(\frac{n\pi}{2019})}.$$

How many distinct elements are in the set of values of S_n as n ranges over all positive integers?

◯ *(A) 1009*

◯ *(B) 2018*

◯ *(C) 2019*

◯ *(D) 4038*

◯ *(E) Infinitely many*

Problem 22 *Let a and b be real numbers so that the equation*

$$x^2 - (a+b)x + 9(a^2+b^2) = \frac{17}{18}$$

has an integer solution. The maximum possible value of $a+4b$ is

◯ *(A)* $\frac{5}{18}$

◯ *(B)* $\frac{5}{9\sqrt{2}}$

◯ *(C)* $\frac{17}{9\sqrt{2}}$

◯ *(D)* $\frac{5\sqrt{34}}{18}$

◯ *(E)* $\frac{17}{3\sqrt{2}}$

Problem 23 *Alice wants to find a collection of distinct numbers in $\{1, 2, 3, 4, \ldots, 15\}$ so that no subset of these numbers sums to exactly 15. What is the largest size of such a collection that can be found?*

◯ *(A) 5*

◯ *(B) 6*

◯ *(C) 7*

◯ *(D) 8*

◯ *(E) 9*

Problem 24 *What is the last non-zero digit of* $2019!$ *?*

◯ *(A) 2*

◯ *(B) 4*

◯ *(C) 5*

◯ *(D) 6*

◯ *(E) 8*

Problem 25 *What is the sum of the coefficients of the odd powers of* x *in* $(1+x+x^2)^7$*?*

◯ *(A) 591*

◯ *(B) 592*

◯ *(C) 1093*

◯ *(D) 1094*

◯ *(E) 2019*