

61st ANNUAL HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 21, 2018
on the campus of the
University of California, San Diego

PART II 4 Questions

Welcome to Part II of the contest!

Please print your Name, School, and Contest ID number:

Name _____
First Last

School _____

3-digit Contest ID number _____

Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number on **each page of the contest**.
2. Part II consists of 4 problems, each worth 25 points. These problems are “essay” style questions. You should put all work towards a solution in the space following the problem statement. You should show all work and justify your responses as best you can.
3. Scoring is based on the progress you have made in understanding and solving the problem. The clarity and elegance of your response is an important part of the scoring. You may use the back side of each sheet to continue your solution, but be sure to call the reader’s attention to the back side if you use it.
4. Give this entire exam to a proctor when you have completed the test to your satisfaction.

Please let your coach know if you plan to attend the Awards Banquet on Wednesday, May 2, 6:00–8:30pm in the UCSD Faculty Club.

Name:

ID Number:

Problem 1 *A convex polygon with n sides has all angles equal to 150 degrees, with the possible exception of one angle. List all the possible values of n .*

Solution: Let x be the unknown angle, where $0 < x < 180$. Then we have $(n - 2)180 = 150(n - 1) + x$ which implies $x = 30n - 210$. Hence $n \in \{8, 9, 10, 11, 12\}$.

Problem 2 Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of positive integers.

- (a) Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function such that $f(f(n)) = n + 2018$, for any $n \in \mathbb{N}$. Prove that $f(n) = n + 1009$, for every $n \in \mathbb{N}$.
- (b) Give an example of function $f : \mathbb{N} \rightarrow \mathbb{N}$ which is not strictly increasing function and satisfies $f(f(n)) = n + 2018$, for every $n \in \mathbb{N}$.

Solution: (a) Since $f(n + 2018) = f(f(f(n))) = f(n) + 2018$, for every $n \in \mathbb{N}$, we get that $f(1 + 2018k) = f(1) + 2018k$, for every non-negative integer k . Let $n \in \mathbb{N}$. Then we can find a non-negative integer k such that $n \leq 1 + 2018k$. Since f is strictly increasing, we have $f(a) - a \leq f(b) - b$, whenever $a \leq b$. This implies that

$$f(1) - 1 \leq f(n) - n \leq f(1 + 2018k) - (1 + 2018k) = f(1) - 1,$$

and thus $f(n) - n = f(1) - 1$, for all $n \in \mathbb{N}$. Hence $f(f(n)) = n + 2(f(1) - 1)$, for all $n \in \mathbb{N}$. This implies that $f(1) - 1 = 1009$ and the conclusion follows.

(b) Define $f(n) = n + 1$, if n odd, and $f(n) = n + 2017$, if n even.

Problem 3 Let A, B, C be points on a circle of radius 1 such that $|AB|^2 + |BC|^2 + |CA|^2 = 8$. Prove that ABC is a right triangle.

Solution: Let O be the center of the circle. Denote by α and β the angles AOB and AOC , respectively. Then the cosine law gives that $|AB|^2 = 2 - 2\cos\alpha$, $|AC|^2 = 2 - 2\cos\beta$, and $|BC|^2 = 2 - 2\cos(\alpha + \beta)$. The hypothesis implies that $\cos\alpha + \cos\beta + \cos(\alpha + \beta) = -1$. Using the sum formula for cosine, we get $(\cos\alpha + 1)(\cos\beta + 1) = \sin\alpha\sin\beta$. Squaring this relation and denoting $x = \cos\alpha, y = \cos\beta$, we get that $(1+x)^2(1+y)^2 = (1-x^2)(1-y^2)$.

Thus, we have that either $(1+x)(1+y) = 0$ or $(1+x)(1+y) = (1-x)(1-y)$. Equivalently, either $x = -1, y = -1$, or $x + y = -1$. This shows that either $\cos\alpha = -1, \cos\beta = -1$, or $\cos(\alpha + \beta) = x + y = -1$, which implies the conclusion.

Problem 4 Define the Collatz sequence starting with a number m to be the sequence of integers defined by $a_1 = m$ and for $n \geq 1$,

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even.} \\ 3a_n + 1 & \text{if } a_n \text{ is odd.} \end{cases}$$

Prove that there are integers n, m with $2^n > m > 0$ so that the Collatz sequence starting from m contains terms larger than 3^{n+2018} .

Solution: We first note that if $a_i = 2 \cdot k - 1$, then $a_{i+2} = 3 \cdot k - 1$. From this, we conclude that if $a_i = 2^r \cdot k - 1$, then $a_{i+2r} = 3^r \cdot k - 1 \geq a_i(3/2)^r$.

We next claim that there is some value of n so that $3^n \equiv -7 \pmod{2^{10000}}$. For this we prove by induction on $\ell > 3$ that there is a power of 3 congruent to $-7 \pmod{2^\ell}$, and a power of 3 congruent to $2^{\ell-1} + 1 \pmod{2^\ell}$. For $\ell = 4$, we may take $3^2 = 9 \equiv -7 \equiv 8 + 1 \pmod{16}$. Assuming that $3^a \equiv 2^{\ell-1} + 1 \pmod{2^\ell}$ and $3^b \equiv -7 \pmod{2^\ell}$, we note that $3^{2a} = (3^a)^2 = (1 + 2^{\ell-1} \cdot q)^2 = 1 + 2^\ell q + 2^{2\ell-2} q^2$ for some odd number q . Therefore, $3^{2a} \equiv 2^\ell + 1 \pmod{2^{\ell+1}}$. For the other conclusion, 3^b is either congruent to $-7 \pmod{2^{\ell+1}}$ or congruent to $-7 + 2^\ell$. In the former case, 3^b works, in the latter $3^{b+2a} \equiv (-7 + 2^\ell)(1 + 2^\ell) \equiv -7 \pmod{2^{\ell+1}}$. This completes the inductive hypothesis, and proves our statement.

We take $m = 2^n - 1$ for an n with $3^n \equiv -7 \pmod{2^{10000}}$.

We note that $a_1 = 2^n - 1$, so by the above $a_{1+2n} = 3^n - 1$. This is congruent to $-8 \pmod{2^{10000}}$, and so we have that $a_{4+2n} = \frac{3^n - 1}{8} \equiv -1 \pmod{2^{9997}}$. Since this is one less than a multiple of 2^{9997} , we find that $a_{4+2n+2 \cdot 9997} = \left(\frac{3^n + 7}{8}\right) (3/2)^{9997} - 1 \geq 3^n / 83^{3000} > 3^{n+2018}$. This completes the proof.