61st ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 21, 2018
on the campus of the
University of California, San Diego

PART I
25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print (**clearly**) your Name and Contest ID number at the top of **each page of the contest**. (Keep your ID number handy for Part II.)
2. Calculators or other electronic devices may **not** be used.
3. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.
4. Clearly indicate your answer in the bubble for each problem. Fill in the bubble, or put a clear X, or a checkmark; just make sure your selection is clear.
5. At the end of Part I, tear off this front page, and turn in your (still-stapled) contest.

Scoring
$+4$ points for a correct answer.
0 points for no answer.
-1 points for an incorrect answer.

Good Luck!

There will be a 15 minute break after Part I before proceeding to Part II.

Please let your coach know if you plan to attend the Awards Banquet on Wednesday, May 2, 6:00–8:30pm in the UCSD Faculty Club.

Name: ID Number:

Problem 1 *What is the largest area of a triangle that can be inscribed in a unit square?*

◯ *(A)* $1/3$

◯ *(B)* $1/2$

◯ *(C)* $1/\sqrt{2}$

◯ *(D)* $2/3$

◯ *(E)* 1

Problem 2 *What is the area of a triangle whose sides have length 10, 13, and 13?*

◯ *(A) 30*

◯ *(B) 50*

◯ *(C) 60*

◯ *(D)* $60\sqrt{2}$

◯ *(E)* $50\sqrt{3}$

Problem 3 *The sum of three numbers is 17. The first is 2 times the second. The third is 5 more than the second. What is the value of the largest of the three numbers?*

◯ *(A) 4*

◯ *(B) 5*

◯ *(C) 6*

◯ *(D) 7*

◯ *(E) 8*

Problem 4 *What is the coefficient of* x^6 *in the expansion of* $(2x^2 - \frac{1}{x})^6$*?*

◯ *(A) 100*

◯ *(B) 140*

◯ *(C) 200*

◯ *(D) 240*

◯ *(E) 260*

Problem 5 *What is the largest integer n such that $20!$ is divisible by 80^n? (Note $20! = 1 \cdot 2 \cdot 3 \cdots 20$).*

◯ *(A) 2*

◯ *(B) 3*

◯ *(C) 4*

◯ *(D) 5*

◯ *(E) 6*

Problem 6 *A rectangle has area 60 and diagonal 13.*
What is the length of its longest side?

◯ *(A)* 5

◯ *(B)* $\sqrt{7}$

◯ *(C)* 7

◯ *(D)* 12

◯ *(E)* 17

Problem 7 *Let n be a positive integer.*
What is the number of ordered pairs (A, B) of disjoint subsets of $\{1, 2, ..., n\}$?

◯ *(A)* 2^n

◯ *(B)* $n2^n$

◯ *(C)* 3^n

◯ *(D)* 4^n

◯ *(E)* $n!$

Problem 8 *One percent milk has 1600 calories per gallon. Four percent milk has 2500 calories per gallon. If the two are mixed to produce two percent milk, how many calories per gallon will this have?*

◯ *(A) 1800*

◯ *(B) 1900*

◯ *(C) 2050*

◯ *(D) 2200*

◯ *(E) 2320*

Problem 9 *Ten people run around a track in the same direction. The fastest of them, Spee D. Pants runs* 10% *faster than the average speed of the others. As the runners travel around the track many times, on average how many times per lap does Spee pass another runner?*

◯ *(A)* $1/10$

◯ *(B)* $9/11$

◯ *(C)* $9/10$

◯ *(D)* $10/11$

◯ *(E)* 1

Problem 10 *Find the number of quadruples* (x, y, z, t) *of pairwise distinct positive integers such that*

$$x^y + x^z + x^t = 31x^{31}.$$

◯ *(A) 6*

◯ *(B) 10*

◯ *(C) 12*

◯ *(D) 18*

◯ *(E) 24*

Problem 11 *Which of the following is the median number in the list?*

◯ *(A)* 20^{18}

◯ *(B)* 201^8

◯ *(C)* 2^{81}

◯ *(D)* 2^{108}

◯ *(E)* 8^{21}

Problem 12 *Find the number of solutions* $x \in [0, 2\pi)$ *of the following trigonometric equation*

$$\cos x \cdot \cos(2x) \cdot \ldots \cdot \cos(2^{2018}x) = \frac{1}{2^{2019}}.$$

◯ *(A)* 2^{2019}

◯ *(B)* $2^{2020} - 2$

◯ *(C)* $2^{2020} - 1$

◯ *(D)* 2^{2020}

◯ *(E)* $2^{2020} + 1$

Problem 13 *Let* $f : (0, \infty) \to \mathbb{R}$ *be a function such that* $f(xy) = f(x)f(y)$*, for all* $x, y > 0$. *Assume that* $f(18) = 12$ *and* $f(24) = 54$.
What is the value of $f(6)$*?*

◯ *(A) 3*

◯ *(B) 6*

◯ *(C) 8*

◯ *(D) 9*

◯ *(E) 12*

Problem 14 *Simplify the expression* $\frac{1}{\sqrt{3-\sqrt{5}}} + \frac{1}{\sqrt{3+\sqrt{5}}}$.

◯ *(A)* $\sqrt{\frac{5}{2}}$

◯ *(B)* $\sqrt{3}$

◯ *(C)* 2

◯ *(D)* $\sqrt{3} + \sqrt{5}$

◯ *(E)* $\sqrt{13}$

Problem 15 *Alice and Bob are playing a game. They start with a pile of 2018 stones and alternate taking turns removing* $1, 3$ *or* 4 *stones from the pile, with Alice going first. A player loses if the pile is empty at the start of their turn. If both players play perfectly, who will win the game?*

◯ *(A) Bob will win.*

◯ *(B) Alice will win, but only if she removes 1 stone on her first turn.*

◯ *(C) Alice will win, but only if she removes 3 stones on her first turn.*

◯ *(D) Alice will win, but only if she removes 4 stones on her first turn.*

◯ *(E) Alice will win no matter what her first move is.*

Problem 16 *If* $1 + x + x^2 + \cdots + x^{99} = -1$*, which of the following equals* x^{199}*?*

◯ *(A)* x

◯ *(B)* $4x - 4x^2 + x^3$

◯ *(C)* $x^{100} + 2x^{99}$

◯ *(D)* $-2x^{98} - 2x^{97} - \cdots - 2x^2 - x - 6$

◯ *(E)* $1 + 2x + 3x^2 + \cdots + 99x^{98} + 98x^{99} + \cdots + x^{197}$

Problem 17 *Let x, y, z be real numbers such that $x = \sqrt{76 - 2yz}, y = \sqrt{73 - 2xz}$ and $z = \sqrt{76 - 2xy}$. What is the value of $x + y + z$?*

- ◯ *(A)* $\sqrt{43}$
- ◯ *(B)* 10
- ◯ *(C)* 15
- ◯ *(D)* 18
- ◯ *(E)* 21

Problem 18 *What is the cardinality of the set $\{2^m \cdot 3^n \cdot 6^p \mid 1 \leq m, n, p \leq 3\}$?*

- ◯ *(A) 12*
- ◯ *(B) 17*
- ◯ *(C) 19*
- ◯ *(D) 22*
- ◯ *(E) 30*

Problem 19 *Consider a parallelogram $ABCD$ and let K, L be the midpoints of BC and CD. It is known that $\angle BAD = \frac{\pi}{3}$ and that the points A, B, K, L lie on a common circle. Find $\angle ADB$.*

- ◯ *(A)* $\frac{\pi}{6}$
- ◯ *(B)* $\frac{\pi}{4}$
- ◯ *(C)* $\frac{\pi}{3}$
- ◯ *(D)* $\frac{5\pi}{12}$
- ◯ *(E)* $\frac{\pi}{2}$

Problem 20 *Let $a_1, a_2, ..., a_n$ be $n \geq 2$ distinct real numbers such that the sum of any $n - 1$ of them belongs to the set $\{0, 1, ..., n - 1\}$.*
What is the value of the sum $a_1 + a_2 + ... + a_n$?

- ◯ *(A)* $-n$
- ◯ *(B)* 1
- ◯ *(C)* $\frac{n}{2}$
- ◯ *(D)* $\frac{(n-1)n}{2}$
- ◯ *(E)* $\frac{n^2}{6} - \frac{n}{3} + 1$

Problem 21 *How many four digit numbers have their largest and smallest digits differing by at most 2? For example 4232 would count since $4 - 2 \leq 2$, but 4758 would not since $8 - 4 > 2$.*

- ◯ *(A) 455*
- ◯ *(B) 509*
- ◯ *(C) 567*
- ◯ *(D) 648*
- ◯ *(E) 810*

Problem 22 *Consider A_1A_2 a segment of size 1 on a line ℓ. On two different sides of the line ℓ, construct the regular polygons*

$$A_1A_2A_3A_4 \ldots A_n \text{ and } A_1A_2A'_3A'_4 \ldots A'_m.$$

If $A_3A'_3 = 1$, the maximal value of the total number of edges of the two polygons equals:

- ◯ *(A) 6*
- ◯ *(B) 12*
- ◯ *(C) 24*
- ◯ *(D) 35*
- ◯ *(E) 49*

Problem 23 *How many distinct sequences of length 1001 can be formed using each of the letters A, B, C an odd number of times?*

- ◯ *(A)* $\frac{3}{4}(3^{1000} - 1)$
- ◯ *(B)* $\frac{1}{2}(3^{1000} - 1)$
- ◯ *(C)* 3^{1000}
- ◯ *(D)* 3^{999}
- ◯ *(E)* $\frac{1}{4}(3^{1000} - 1)$

Problem 24 *How many degree* 4 *polynomials with coefficients in the set* $\{1, 2, \ldots, n\}$ *are divisible by* $x^2 + x + 1$*?*

- ◯ *(A)* $\frac{n^3-n}{6}$
- ◯ *(B)* $\frac{n(n-1)(2n-1)}{6}$
- ◯ *(C)* $\frac{5^{n-1}-1}{2}$
- ◯ *(D)* $\frac{n(n-1)}{2}$
- ◯ *(E)* $\frac{5^n-1}{4}$

Problem 25 *The triangle* ABC *has sides*

$$AB = \sqrt{3}, \;\; BC = \sqrt{3}, \;\; AC = \sqrt{5}.$$

In the interior of the triangle consider a point D *such that*

$$AD^2 = CD^2 = BD^2 + 2.$$

The length of CD *equals*

- ◯ *(A)* $\frac{6\sqrt{2}}{7}$
- ◯ *(B)* $\frac{3\sqrt{2}}{5}$
- ◯ *(C)* $\frac{4\sqrt{3}}{7}$
- ◯ *(D)* $\sqrt{\frac{15}{7}}$
- ◯ *(E)* $\sqrt{5}$