60th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 22, 2017
on the campus of the
University of California, San Diego

PART II
4 Questions

Welcome to Part II of the contest!

Please print your Name, School, and Contest ID number:

Name ____
First Last

School ____

3-digit Contest ID number ____________________________

Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

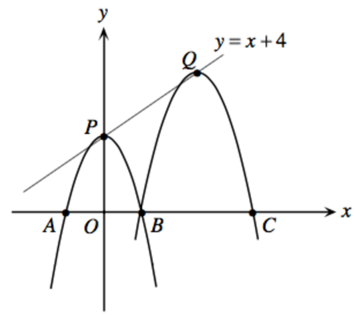
1. Part II consists of 4 problems, each worth 25 points. These problems are "essay" style questions. You should put all work towards a solution in the space following the problem statement. You should show all work and justify your responses as best you can.

2. Scoring is based on the progress you have made in understanding and solving the problem. The clarity and elegance of your response is an important part of the scoring. You may use the back side of each sheet to continue your solution, but be sure to call the reader's attention to the back side if you use it.

3. Give this entire exam to a proctor when you have completed the test to your satisfaction.

Please let your coach know if you plan to attend the Awards Dinner on Wednesday, May 3, 6:00–8:30pm in the UCSD Faculty Club.

1. The parabola with equation $y = 4 - x^2$ has vertex P and intersects the x-axis at A and B. The parabola is translated from its original position so that its vertex moves along the line $y = x + 4$ to the point Q. In this position, the parabola intersects the x-axis at B and C. Determine the coordinates of C.



Solution. The original parabola has equation $y = (2 - x)(2 + x)$; its x-intercepts are therefore at $A = (-2, 0)$ and $B = (2, 0)$ while its vertex is at $P = (0, 4)$, which is on the line $y = x + 4$. The translation of the parabola is thus of the form $(x, y) \mapsto (x - t, y - t)$ for some t, since the line has slope 1. Thus, the new parabola has equation

$$y - t = -(x - t)^2 + 4.$$

Since the point $B = (2, 0)$ is on this parabola, we can solve for t: $0 - t = -(2 - t)^2 + 4$, which simplifies to $t^2 - 5t = 0$, whose solutions are $t = 0$ and $t = 5$. The choice $t = 0$ yields the original parabola; thus, the translated one has $t = 5$, giving the parabola $y - 5 = -(x - 5)^2 + 4$, or $y = 9 - (x - 5)^2 = (8 - x)(x - 2)$. The x-intercepts are thus at $x = 2$ and $x = 8$, and so $C = (8, 0)$.

2. 15 pairwise coprime integers are chosen from the set $\{2, \ldots, 2017\}$. Show that at least one prime number was chosen.

Solution. Denote by $a_1, \ldots, a_{15}$ the 15 integers. We have

$$a_i \leq 2017, \quad \text{for all } 1 \leq i \leq 15.$$

We assume for a contradiction that none of $a_1, \ldots, a_{15}$ are prime. Then, for $1 \leq i \leq 15$, a_i must be divisible by the product $p_i q_i$ of two primes, not necessarily distinct. In particular

$$p_i q_i \leq a_i$$

Without loss of generality, we may suppose $p_i \leq q_i$. Therefore

$$p_i^2 \leq p_i q_i \leq a_i \implies p_i \leq \sqrt{a_i} \leq \sqrt{2017} \implies p_i \leq 44.$$

Since the a_i's are pairwise coprime, it follows that the p_i's are distinct. Thus $\{p_1, \ldots, p_{15}\}$ is a set of 15 primes less or equal than 44. However, there are only 14 such primes, namely

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43.$$

This is a contradiction, showing that one of the integers $a_1, \ldots, a_{15}$ must have been prime to begin with.

3. Let $a_1 < a_2 < \cdots < a_n$ be n real numbers such that the set

$$A = \{a_j - a_i \mid 1 \leq i < j \leq n\}$$

has exactly $n - 1$ elements. Prove that $a_1, a_2, ..., a_n$ form an arithmetic progression.

Solution. Note that

$$a_2 - a_1 < a_3 - a_1 < \cdots < a_{n-1} - a_1 < a_n - a_1 \quad \text{and}$$

$$a_3 - a_2 < a_4 - a_2 < \cdots < a_n - a_2 < a_n - a_1$$

are two strictly increasing sequences of $(n-1)$ elements of A. Since A has exactly $(n-1)$ elements, the two sequences must be equal. In other words,

$$a_k - a_1 = a_{k+1} - a_2, \quad \text{for every } \ 2 \leq k \leq n-1.$$

Therefore, $a_{k+1} - a_k = a_2 - a_1$, for every $2 \leq k \leq n-1$, which shows that $a_1, a_2, \ldots, a_n$ form an arithmetic progression.

4. Let S be a region in space given as a finite union of unit cubes whose corners have integer coordinates. Let A be the projection of S onto the (x, y)-plane, B the projection onto the (y, z)-plane and C the projection onto the (x, z)-plane. Prove that

$$\mathrm{Vol}(S) \leq \sqrt{\mathrm{Area}(A)\mathrm{Area}(B)\mathrm{Area}(C)}.$$

Solution. Associate each unit cube in S or unit square in $A, B,$ or C to the corner with smallest coordinates. Note that the cube corresponding to (i, j, k) is in S only if the square corresponding to (i, j) is in A, the square corresponding to (j, k) is in B and the square corresponding to (i, k) is in C. Thus, $\mathrm{Vol}(S)$ is at most the number of triples of integers (i, j, k) so that this holds.

For each i, let c_i be the number of integers k so that the square corresponding to (i, k) is in C, and let b_j be the number of k so that the square corresponding to (j, k) is in B. Note that for fixed integers (i, j) the number of integers k so that (i, j, k) satisfy the condition above is at most $\min(b_j, c_i)$, and is only positive if the square corresponding to (i, j) is in A. Therefore, we have

$$\begin{aligned}
\mathrm{Vol}(S)^2 \leq \left(\sum_{(i,j)\in A} \min(b_j, c_i)\right)^2 \leq \left(\sum_{(i,j)\in A} \sqrt{b_j c_i}\right)^2 &\leq \left(\sum_{(i,j)\in A} 1\right)\left(\sum_{(i,j)\in A} b_j c_i\right) \\
&= \mathrm{Area}(A) \sum_{i,j} b_j c_i \\
&= \mathrm{Area}(A) \left(\sum_j b_j\right)\left(\sum_i c_i\right) \\
&= \mathrm{Area}(A)\mathrm{Area}(B)\mathrm{Area}(C).
\end{aligned}$$

The second line above is because $\min(b_j, c_i) \leq \sqrt{b_j c_i}$. The third line is by the Cauchy-Schwarz inequality. The fourth line is because the area of A equals the number of squares in A. The fifth line follows by noting that the sum separates into separate sums over i and j. The last line is because the area of B is the number of squares in B, which is the sum over j of the number of squares with y-coordinate j, or the sum of b_j (a similar argument holds for the area of C).

This completes the proof.