

60th ANNUAL HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 22, 2017
on the campus of the
University of California, San Diego

PART I 25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print your name on the lower right side of the answer sheet.
2. Print and bubble in your 3-digit contest ID number under “EXAM NUMBER”. **Keep your ID number for Part II.**
Do not make extraneous marks on the answer sheet. Your answers must be indicated by neatly blackening the box with a #2 pencil. If you erase, do so thoroughly!
3. Calculators may **not** be used. You should use the exam paper for scratchwork.
4. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.
5. At the end of Part I, hand in the answer sheet — you may keep the examination questions. Make sure your name and ID number are on the answer sheet.

Scoring

- +4 points for a correct answer.
 - 0 points for no answer.
 - 1 points for an incorrect answer.
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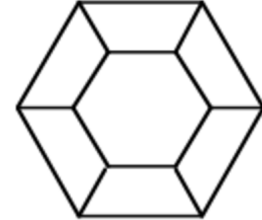
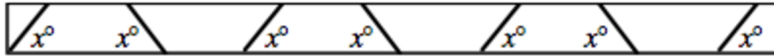
Good Luck!

There will be a 15 minute break after Part I before proceeding to Part II.

Please let your coach know if you plan to attend the Awards Banquet on Wednesday, May 3, 6:00–8:30pm in the UCSD Faculty Club.

You may take these exam questions with you after you are done. You may write on this exam and use it to discuss your results outside the room after completion of the exam.

1. Six identical pieces are cut from a board, as shown in the diagram. The angle of each cut is x° . The pieces are assembled to form a hexagonal picture frame as shown. What is the value of x ?



- (A) 30 (B) 45 (C) 60 (D) 90 (E) 120

2. A circle passes through the origin and the points of intersection of the parabolas $y = x^2 - 3$ and $y = -x^2 - 2x + 9$. Determine the coordinates of the center of this circle.

- (A) (0, 0) (B) (-0.5, 3.5) (C) (-1, 7) (D) (-3.5, 6.5) (E) (2, 1)

3. In an arithmetic sequence with five terms, the sum of the first two terms is 2 and the sum of the last two terms is -22 . What is the third term in the sequence? (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9, 11 is an arithmetic sequence with five terms.)

- (A) -5 (B) -1 (C) 7 (D) $\frac{2}{3}$ (E) 0

4. Let x_1, x_2 be distinct solutions of the equation $x^2 - 4x + 1 = 0$. What is the value of $x_1^3 + x_2^3$?

- (A) -1 (B) 52 (C) 112 (D) $1 + 3\sqrt{15}$ (E) 60

5. A fair six-sided die is rolled 4 times. What is the probability that the sum of the numbers that show up is a multiple of 3?

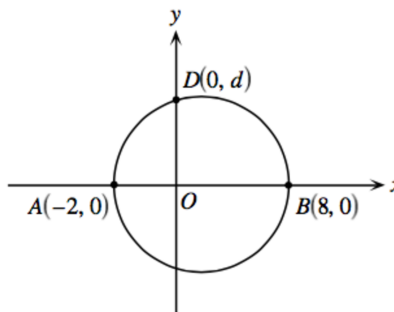
- (A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{6}$ (D) $\frac{1}{9}$ (E) $\frac{3}{10}$

6. Which of the following is the largest?

- (A) 2^{10^4} (B) 201^5 (C) 2017^{2017} (D) $7^{2^{10}}$ (E) $2^{1^{80}}$

7. A circle, with diameter AB as shown, intersects the positive y -axis at point $D(0, d)$. What is the value of d ?

- (A) 1 (B) 2 (C) 3
(D) 4 (E) 5



8. Assume that a, b, c, d, e are real numbers such that

$$ab = 2, \quad bc = 3, \quad cd = 9, \quad de = 8.$$

Find the value of the fraction $\frac{a}{e}$.

- (A) $\frac{1}{4}$ (B) $\frac{3}{4}$ (C) $\frac{3}{8}$ (D) $\frac{1}{72}$ (E) cannot be determined

9. A circle is divided into arcs of lengths 2, 3, 8, x . The center angle of the arc of length 2 equals 30° . What is the value of x ?

- (A) 5 (B) 4π (C) $2\pi - 13$ (D) 35 (E) 11

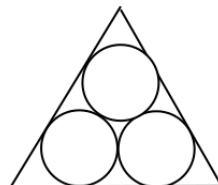
10. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a function such that

$$3f(x) + 4f\left(\frac{6}{x}\right) = x - 1.$$

The value of $f(12)$ is

- (A) 1 (B) $-\frac{2}{7}$ (C) $\frac{3}{2}$ (D) -5 (E) $\frac{5}{7}$
11. Three dice are rolled and their results multiplied together. What is the probability that the resulting number is 20?
- (A) $\frac{1}{36}$ (B) $\frac{1}{24}$ (C) $\frac{1}{20}$ (D) $\frac{1}{18}$ (E) $\frac{1}{6}$
12. A cube has edges of length n , where n is an integer. Three faces, meeting at a corner, are painted red; the other faces are not. The cube is then cut into n^3 smaller cubes of unit length. If exactly 125 of these cubes have no faces painted red, what is the value of n ?
- (A) 3 (B) 4 (C) 5 (D) 6 (E) 7
13. Two points are selected at random on a circle of radius 1. What is the probability that the distance between the points is at most 1?
- (A) $\frac{1}{10}$ (B) $\frac{1}{2\sqrt{3}}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$ (E) $\frac{1}{4}$
14. In the figure below, three congruent and mutually tangent circles are inscribed in an equilateral triangle with sides of length 1. What is the radius of one of the circles?

- (A) $\frac{\sqrt{3}}{6}$ (B) $\frac{\sqrt{3} - 1}{4}$ (C) $\frac{1}{3}$
(D) $\frac{\sqrt{3} + 1}{6}$ (E) $\frac{1}{2} - \frac{1}{\sqrt{3}}$



15. Evaluate the expression

$$\frac{(5+6)(5^2+6^2)(5^4+6^4)\cdots(5^{1024}+6^{1024})+5^{2048}}{3^{1024}}.$$

- (A) 2^{1024} (B) 2^{2048} (C) 6^{1024} (D) 12^{1024} (E) 5^{2048}

16. Let n be a positive integer. What is the number of ordered triples (a, b, c) of integers $a, b, c \in \{1, 2, \dots, n\}$ such that $a + b = c$?

- (A) $\frac{n(n+1)}{2}$ (B) $n!$ (C) n^2 (D) $\frac{(n-1)n}{2}$ (E) 2^{n-1}

17. Let n be a positive integer. What is the number of functions $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n+2\}$ satisfying $f(x) < f(y)$ for every $x, y \in \{1, 2, \dots, n\}$ such that $x < y$?

- (A) $\frac{n(n+1)}{2}$ (B) $\frac{(n+1)(n+2)}{2}$ (C) $\frac{n(n+1)(n+2)}{6}$ (D) $2^n + 1$ (E) n^3

18. Let f be a function which assigns to every real number x a real number $f(x)$. Assume that $f(x+y) = f(x) + f(y) - 5$, for every real numbers x and y , and that $f(1) = 8$. What is the value of $f(2017)$?

- (A) 6056 (B) 2018 (C) 5815 (D) -12 (E) 6520

19. Suppose that for real numbers a, b, c we have that $a + b + c = 1$, $a^2 + b^2 + c^2 = 7$, and $a^3 + b^3 + c^3 = 13$. What is $\min\{a, b, c\}$?

- (A) -2 (B) 2 (C) $1 - \sqrt{2}$ (D) 0 (E) -1

20. How many points with integer coordinates are (strictly) inside the triangle defined by the lines $x = 0$, $y = 0$ and $5x + 8y = 40$?

- (A) 12 (B) 13 (C) 14 (D) 21 (E) 28

21. The set $\{1, 2, \dots, 999\}$ is partitioned into n disjoint subsets A_1, \dots, A_n such that:
- each subset A_i contains at least 2 elements
 - the sum of any two distinct elements in the same set A_i is not divisible by n .

What is the smallest possible value of n ?

- (A) 2 (B) 31 (C) 32 (D) 33 (E) 333

22. Let a, b, c, d be positive integers such that

$$\log_a b = \frac{3}{2}, \quad \log_c d = \frac{5}{4}, \quad a - c = 9.$$

Find the value of the expression $b - d$.

- (A) 93 (B) 23 (C) -3 (D) 1 (E) cannot be determined

23. Find the number of integer solutions of the system

$$xy + zt = -2, \quad xz + yt = -2, \quad xt + yz = -2.$$

- (A) 16 (B) 8 (C) 4 (D) 0 (E) cannot be determined

24. Let X be a point inside a rectangle $ABCD$. Assume that $|AB| = |AX| = 5$, $|BX| = \sqrt{10}$, and $|DX| = \sqrt{17}$. What is $|BC|$?

- (A) 1 (B) 5 (C) $\sqrt{7}$ (D) 2 (E) 4

25. Let N be the number of ways the set $\{1, 2, \dots, 2017\}$ can be partitioned into disjoint nonempty subsets. What is the remainder obtained by dividing N by 2017?

- (A) 0 (B) 1 (C) 2 (D) 4 (E) 2016