59th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 16, 2016
on the campus of the
University of California, San Diego

PART II: Solutions
4 Questions

1. A fair coin is tossed 10 times. What is the probability that it comes up heads 5 times in a row, but not 6 times in a row?

Solution. Since there are no 6 heads in a row, the configuration of tosses must look like one of the following:

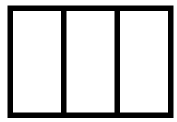
$$HHHHHTABCD$$
$$THHHHHTABC$$
$$ATHHHHHTBC$$
$$ABTHHHHHTC$$
$$ABCTHHHHHT$$
$$ABCDTHHHHH$$

where A, B, C, D can be heads H or tails T independently. All 6 of these types of configurations are distinct: there is no way to select the values of A, B, C, D so that two of them actually match up. So to count the number of configuration, we just need to count the number of each type. As A, B, C, D can each take the two values H or T independently, the first and sixth patterns each account for $2^4 = 16$ configurations, while the middle 4 each account for $2^3 = 8$ configurations. So there are $2 \cdot 16 + 4 \cdot 8 = 64 = 2^6$ configurations having a string of 5 heads, but no string of 6 heads. This is out of 2^{10} total possible configurations; so the probability is

$$\frac{2^6}{2^{10}} = \frac{1}{2^4} = \frac{1}{16}.$$

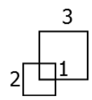
2. Let R be a rectangle whose sides have lengths 2 and 3. Choose any four points inside R. Prove that there exist two of these points whose distance from each other is less than $\sqrt{5}$.

Solution. Divide R into three 1×2 rectangles, as shown in the picture below:



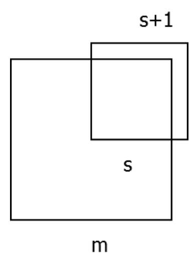
By the pigeonhole principle, we can find two of the four points inside or on the sides of one of the rectangles, called S. The distance between these points is at most $\sqrt{5}$ (the length of the diagonal of S). The distance cannot be equal to $\sqrt{5}$, since this would force the points to be opposite vertices of S, contradicting that the points lie inside R.

3. What is the largest positive integer n so that n is not the area of a union of two squares with corners on lattice points and sides parallel to the x- and y-axes? For example, 12 is the area of the following union of squares of side lengths 2 and 3.

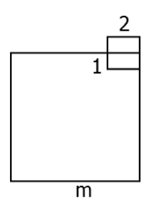


Solution. The answer is 3. Such a union cannot have area 3 since if either square has side length 2 or more that is already too much, and otherwise the total area is at most $1 + 1 = 2 < 3$.

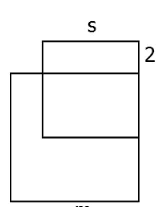
The more difficult problem is showing that any $n > 3$ can be written as such an area. Let m be the largest integer so that $m^2 \leq n$. Note that $m \geq 2$ and that $n = m^2 + k$ for some integer $0 \leq k \leq 2m$. We split into cases based on whether k is even or odd. If k is odd, we write $k = 2s + 1$, and note that n is the area of the union of the following pair of squares of side lengths m and $s + 1$ overlapping in a square of side length s.



If k is even, we need a few more cases. Firstly if $k = 0$, we write n as the area of union of a square of side length m with itself. If $k = 2$, we write it as the union of a square of side length 2 with one of side length m that overlap in a 1×2 rectangle, as show below:



Finally, if $k > 2$ is even, we write $k = 2s$ with $s \leq m$. We write n as the union of a square of side length n with a square of side length s that overlap in an $s \times (s - 2)$ rectangle as shown below:



4. Let $p_1 = 2$ and define p_{n+1} to be the largest prime divisor of $1 + p_1 p_2 \dots p_n$. Is 11 a term in the sequence $\{p_n\}$?

Solution. 11 is not a term in the sequence.

For a contradiction, assume $p_N = 11$ for some index N. By direct calculation we see that

$$p_1 = 2, p_2 = 3, p_3 = 7, p_4 = 43,$$

so $N \geq 5$. Let

$$A = 1 + p_1 p_2 p_3 \dots p_{N-1}.$$

In particular

$$A = 1 + 2 \cdot 3 \cdot 7 \cdot B = 1 + 42B$$

for $B = p_4 \dots p_{N-1}$. This shows that A cannot have $2, 3, 7$ as prime factors. Since by assumption 11 is the largest prime dividing A, 5 and 11 are the only possible prime factors of A. Therefore,

$$A = 5^k \cdot 11^\ell,$$

for integers $k, \ell \geq 0$. We reduce A modulo $3, 4$ and 7 to derive a contradiction:

(i) We begin by claiming that p_n is odd for all $n > 1$. Indeed, since $p_1 = 2$, it follows that $1 + p_1 \dots p_{n-1}$ is odd, and thus $p_n \neq 2$ being a divisor of $1 + p_1 \dots p_{n-1}$. As a consequence,

$$A = 1 + 2p_2 \dots p_{N-1} = 1 + 2 \cdot \text{ odd number}$$

so $A \equiv 3 \mod 4$. Thus

$$5^k \cdot 11^\ell \equiv 3 \mod 4 \implies 1 \cdot (-1)^\ell \equiv -1 \mod 4 \implies \ell \text{ is odd.}$$

(ii) Since $A = 1 + 42B \implies A \equiv 1 \mod 3$. Reducing modulo 3, we have

$$5^k \cdot 11^\ell \equiv 1 \mod 3 \implies (-1)^k \cdot (-1)^\ell \equiv 1 \mod 3 \implies k + \ell \text{ is even.}$$

As a result of (i) and (ii), both k and ℓ must be odd.

(iii) Finally, $A = 1 + 42B \implies A \equiv 1 \mod 7$, hence reducing modulo 7 we have

$$5^k \cdot 11^\ell \equiv 1 \mod 7 \implies (-2)^k \cdot 4^\ell \equiv 1 \mod 7 \implies 2^{k+2\ell} \equiv (-1)^k \mod 7$$
$$\implies 2^{k+2\ell} \equiv -1 \mod 7,$$

using the already established fact that k is odd.

However, by examining the remainders $2^m \mod 7$, for positive integers m, we only obtain the values $\{1, 2, 4\}$. This means the value $-1 \mod 7$ is not possible, thus obtaining a contraction.

To see the last claim about $2^m \mod 7$, write $m = 3q + r$ for $r \in \{0, 1, 2\}$. Then

$$2^m = 2^{3q+r} = 8^q \cdot 2^r \equiv 1 \cdot 2^r \mod 7$$

and note that $2^r \in \{1, 2, 4\}$ if $r = 0, 1, 2$.