59th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 16, 2015
on the campus of the
University of California, San Diego

PART I
25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print your name on the lower right side of the answer sheet.
2. Print and bubble in your 3-digit contest ID number under “EXAM NUMBER”. **Keep your ID number for Part II**.
   Do not make extraneous marks on the answer sheet. Your answers must be indicated by neatly blackening the box with a #2 pencil. If you erase, do so thoroughly!
3. Calculators may **not** be used. You should use the exam paper for scratchwork.
4. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.
5. At the end of Part I, hand in the answer sheet — you may keep the examination questions. Make sure your name and ID number are on the answer sheet.

Scoring
$+4$ points for a correct answer.
0 points for no answer.
-1 points for an incorrect answer.

Good Luck!

There will be a 15 minute break after Part I before proceeding to Part II.

Please let your coach know if you plan to attend the Awards Banquet on Wednesday, April 27, 6:00–8:30pm in the UCSD Faculty Club.

You may take these exam questions with you after you are done. You may write on this exam and use it to discuss your results outside the room after completion of the exam.

1. How many three digit numbers have all three digits distinct?

   (A) 899 (B) 810 (C) 720 (D) 648 (E) 504

2. If $x + \frac{1}{x} = 3$, what is the value of $x^3 + \frac{1}{x^3}$?

   (A) 3 (B) 9 (C) 18 (D) 27 (E) 36

3. Let S be a square and T be an equilateral triangle, and suppose S and T have the same perimeter. What is the ratio of their areas $\frac{\text{area}(S)}{\text{area}(T)}$?

   (A) 2 (B) $\sqrt{3}$ (C) $\frac{2\sqrt{3}}{3}$ (D) $\frac{3\sqrt{3}}{4}$ (E) $\sqrt{\frac{3}{2}}$

4. A round table has six seats, labeled 1 through 6 in clockwise order. You are trying to come up with a seating arrangement for six people, three women (Anna, Bella, and Chloe) and three men (David, Evan, and Fred), so that each person sits next to at least one other person of the same sex. How many possible seating charts are there?

   (A) 6 (B) 144 (C) 216 (D) 120 (E) 720

5. Alice and Bob are playing a game. Each player starts with a pile of tokens. When it is a player's turn, they count how many tokens are in their pile, and attempt to remove and discard that many tokens from their opponent's pile. If there are not enough tokens in their opponent's pile to do this, the current player wins; otherwise, the turn is finished and their opponent goes next. On Alice's turn, her pile has 13 tokens. What is the smallest number of tokens Bob must have in his pile in order to eventually win?

   (A) 21 (B) 22 (C) 23 (D) 24 (E) 25

6. A quadratic polynomial p satisfies $p(1) = 1$, $p(2) = 0$, and $p(3) = 2$. What is $p(5)$?

   (A) -11 (B) -5 (C) 0 (D) 9 (E) 15

7. Let a, b, and x be real numbers such that $x > a^2$ and $x > b^2$. Suppose that $\sqrt{x-b^2} - \sqrt{x-a^2} = a - b$. What is the value of $\sqrt{x-b^2} + \sqrt{x-a^2}$?

   (A) $a + b$ (B) a (C) b (D) $\sqrt{a^2+b^2}$ (E) cannot be determined

8. Simplify $(\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7})(\sqrt{5}-\sqrt{6}+\sqrt{7})(-\sqrt{5}+\sqrt{6}+\sqrt{7})$.

   (A) $\sqrt{210}$ (B) 0 (C) 12 (D) $4+2\sqrt{30}$ (E) 104

9. How many real solutions does the equation $x^6 - 3x^2 + 1 = 0$ have?

   (A) 0 (B) 1 (C) 2 (D) 4 (E) 6

10. A fair six-sided die is rolled n times. What is the probability that the sum of the numbers that show up is equal to $n + 2$?

    (A) $\frac{n}{6^n}$ (B) $\frac{1}{n}$ (C) $\frac{n+2}{4^n}$ (D) $\frac{\binom{n+1}{2}}{6^n}$ (E) $\frac{n^2}{6^n}$

11. Let ABC be an equilateral triangle with side length 1. Let L be a line in the plane of ABC. What is the smallest possible value of the sum of the distances of A, B, and C to L?

    (A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$ (E) 2

12. Let n be a positive integer. How many functions $f\colon \{1,2,\ldots,2n\} \to \{1,2,\ldots,2n\}$ satisfy, for every $x \in \{1,2,\ldots,2n\}$, both $f(f(x)) = x$ and $f(x) \neq x$?

    (A) $1\cdot 3\cdots(2n-1)$ (B) $2\cdot 4\cdots 2n$ (C) $2^{2n}-1$ (D) n^2+n-1 (E) $n!$

13. Let n be a positive integer. How many pairs of positive integers a, b satisfy $a^2 - b^2 = 4^n$?

(A) $n-1$ (B) n (C) $n+1$ (D) n^2 (E) 2^n

14. What are the last three decimal digits of 3^{2016}?

(A) 721 (B) 243 (C) 521 (D) 035 (E) 871

15. The sum of the first n positive integers is a three-digit number, all of whose digits are equal. What is the sum of the digits of n?

(A) 5 (B) 6 (C) 9 (D) 12 (E) 15

16. Each cell of a 10×11 array is filled with a real number such that the rows and columns form arithmetic progressions. The sum of the numbers in the 4 corners of the array is 50. What is the sum of the numbers in all cells of the array?

(A) 750 (B) 1375 (C) 3000 (D) 5500 (E) cannot be determined

17. Let $x_1, x_2, \ldots, x_{2016}$ be real numbers satisfying the following system of equations:

$$\begin{aligned} 2x_1 + x_2 + x_3 + \cdots + x_{2016} &= 1 \\ x_1 + 2x_2 + x_3 + \cdots + x_{2016} &= 2 \\ x_1 + x_2 + 2x_3 + \cdots + x_{2016} &= 3 \\ &\vdots \\ x_1 + x_2 + x_3 + \cdots + 2x_{2016} &= 2016. \end{aligned}$$

What is the smallest n for which $x_n > 0$?

(A) 1005 (B) 1006 (C) 1007 (D) 1008 (E) 1009

18. For what integer n is $\frac{1}{n}$ closest to $\sqrt{1{,}000{,}000} - \sqrt{999{,}999}$?

(A) 1995 (B) 1999 (C) 2000 (D) 2016 (E) 2017

19. Two circles C_1 and C_2 have radii 1 and 3, and their centers are 10 units apart. Which of the following best describes the locus of points that are midpoints between some point on C_1 and some point on C_2?

(A) circle (B) line segment (C) disk (D) ellipse (E) annulus

20. Define the recursive sequence $a_{n+2} = a_n + a_{n+1}$ with $a_1 = 3$ and $a_2 = 7$. What is the remainder obtained by dividing $a_1^4 + \cdots + a_{2016}^4$ by 16?

(A) 0 (B) 1 (C) 2 (D) 4 (E) 15

21. If $\sqrt[n]{29\sqrt{2}+41} - \sqrt[n]{29\sqrt{2}-41} = 2$, the value of n is

(A) 2 (B) 3 (C) 4 (D) 5 (E) 7

22. Let $a_1, a_2, \ldots, a_{2016}$ be real numbers satisfying

$$a_1 = 0, \quad |a_2| = |a_1 + 1|, \quad |a_3| = |a_2 + 1|, \quad \ldots \quad |a_{2016}| = |a_{2015} + 1|.$$

What is the smallest possible value that the average $\dfrac{a_1 + \cdots + a_{2016}}{2016}$ can take?

(A) 0 (B) $-\frac{1}{2}$ (C) -1 (D) -2016 (E) -2017

23. Determine the remainder when $3^{2^{2016}} - 1$ is divided by 2^{2019}.

(A) 2^{2016} (B) 2^{2017} (C) 2^{2018} (D) $2^{2^{2016}}$ (E) 2016^2

24. Seven equally-spaced points are selected on a circle. If three are chosen randomly, what is the probability that the triangle they determine is acute?

(A) $\frac{1}{7}$ (B) $\frac{2}{5}$ (C) $\frac{3}{5}$ (D) $\frac{5}{7}$ (E) $\frac{42}{35}$

25. Consider the recursive sequence

$$x_{n+1} = 4x_n(1 - x_n), \qquad x_0 = a.$$

For how many values of a is it true that $x_{2016} = 0$?

(A) 2 (B) 2^{2015} (C) $2^{2015} + 1$ (D) $2^{2016} + 1$ (E) infinitely many