# 58th ANNUAL
# HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 18, 2015
on the campus of the
University of California, San Diego

**PART II**
**4 Questions**

**Welcome to Part II of the contest!**

Please print your Name, School, and Contest ID number:

Name ______________________________________________
First                                    Last

School ______________________________________________

3-digit Contest ID number ________________________________

**Please do not open the exam until told do so by the proctor.**

EXAMINATION DIRECTIONS:

1. Part II consists of 4 problems, each worth 25 points. These problems are "essay" style questions. You should put all work towards a solution in the space following the problem statement. You should show all work and justify your responses as best you can.

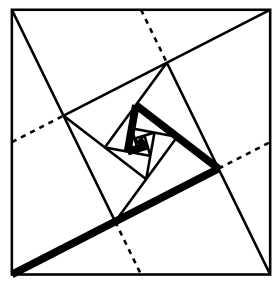
2. Scoring is based on the progress you have made in understanding and solving the problem. The clarity and elegance of your response is an important part of the scoring. You may use the back side of each sheet to continue your solution, but be sure to call the reader's attention to the back side if you use it.

3. Give this entire exam to a proctor when you have completed the test to your satisfaction.

**Please let your coach know if you plan to attend the Awards Dinner on Wednesday, April 29, 6:00–8:15pm in the UCSD Faculty Club**.

1. Let $a, b$ be integers such that $|1 + ab| < |a + b|$. Prove that either $a$ or $b$ is equal to $0$.

2. The fractal below is formed by starting with an outer square of edge length 1. Lines are drawn between each vertex of this square and the midpoint of the side two steps counterclockwise of it. These four lines form a smaller square in the middle and this process in repeated on this square, producing a new square on which the process is repeated and so on. A spiral is then drawn as shown connecting a vertex of each square to a corresponding vertex of the next smaller one. What is the total length of the spiral?



3. Let $n$ be a positive integer. Determine all functions $f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ which satisfy $|f(a) - f(b)| \geqslant |a - b|$, for all $a, b \in \{1, 2, ..., n\}$.

4. Consider the set $\mathcal{P}$ of all points in the plane with integer coordinates. First, two points $A, B$ of $\mathcal{P}$ are painted black. A new point in $\mathcal{P}$ is painted black provided it lies on some circle of rational radius passing through two other black points.

   Show that either infinitely many points of $\mathcal{P}$ will be painted black, or else no point of $\mathcal{P}$ other than $A$ and $B$ will be painted black. Furthermore, show that both situations can occur.