58th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

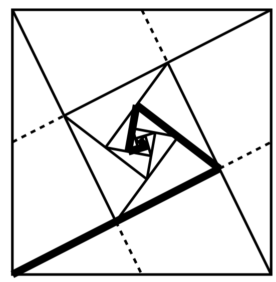
April 18, 2015
on the campus of the
University of California, San Diego

PART II
SOLUTIONS

1. Let a, b be integers such that $|1 + ab| < |a + b|$. Prove that either a or b is equal to 0.

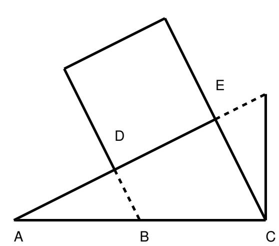
   Solution. Assume that $|1 + ab| < |a + b|$. Then $(1 + ab)^2 < (a + b)^2$ and thus $1 + a^2b^2 - a^2 - b^2 < 0$. This can be rewritten as $(1 - a^2)(1 - b^2) < 0$. Hence, one of$1 - a^2$ and $1 - b^2$ is > 0 and the other is < 0. For the sake of argument, suppose it is a which satisfies $1 - a^2 > 0$. Thus $a^2 < 1$. Since a is an integer, so is a^2, and thus $a = 0$.

2. The fractal below is formed by starting with an outer square of edge length 1. Lines are drawn between each vertex of this square and the midpoint of the side two steps counterclockwise of it. These four lines form a smaller square in the middle and this process in repeated on this square, producing a new square on which the process is repeated and so on. A spiral is then drawn as shown connecting a vertex of each square to a corresponding vertex of the next smaller one. What is the total length of the spiral?



Solution. The answer is $\frac{1+\sqrt{5}}{2}$.

Consider the diagram below consisting of an edge AC of one of the squares in our figure and a corresponding edge DE of the next smaller square. Let B be the midpoint of AC. Since the lines CE and BD run along opposite sides of the small square, they must be parallel. Therefore since BD bisects AC, it must also bisect AE, thus $AD = DE$. By rotational symmetry it must be the case that $AD = EC$. Consider the right triangle AEC. The legs have length DE and $2DE$, therefore by the Pythagorean Theorem, we have that $AC = \sqrt{5}DE$. Therefore the ratio of side lengths of consecutive squares in the figure is $\sqrt{5}$.



Next suppose that AC were an edge of the original square and AE the first segment in the spiral. We have by the above that $AE = 2DE = \frac{2}{\sqrt{5}}AC = \frac{2}{\sqrt{5}}$. Thus, the length of the first segment is $\frac{2}{\sqrt{5}}$. The length of each subsequent segment is small than the previous by a factor of $\sqrt{5}$. Therefore, the total length of the spiral is

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2}{\sqrt{5}^{n+1}} &= \frac{2/\sqrt{5}}{1-1/\sqrt{5}} \\ &= \frac{2}{\sqrt{5}-1} \\ &= \frac{2(1+\sqrt{5})}{5-1} \\ &= \frac{1+\sqrt{5}}{2}.\end{aligned}$$

3. Let n be a positive integer. Determine all functions $f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ which satisfy $|f(a) - f(b)| \geqslant |a - b|$, for all $a, b \in \{1, 2, ..., n\}$.

Solution. Putting $a = n$ and $b = 1$, we get that $|f(n) - f(1)| \geqslant n - 1$. Note that if $a, b \in \{1, 2, ..., n\}$ then $|a - b| \geqslant n - 1$ if and only if $\{a, b\} = \{1, n\}$. We deduce that either (1) $f(n) = n$ and $f(1) = 1$, or (2) $f(n) = 1$ and $f(1) = n$.

Assume first that we are in case (1), and let $1 \leqslant a \leqslant n$. Then by applying the given inequality to the pairs $(a, 1)$ and (a, n), we get that $|f(a) - 1| \geqslant a - 1$ and $|f(a) - n| \geqslant (n - a)$. Since $1 \leqslant f(a) \leqslant n$, we can rewrite these inequalities as $f(a) \geqslant a$ and $f(a) \leqslant n - (n - a) = a$. Thus, $f(a) = a$, for all $a \in \{1, 2, ..., n\}$.

In case (2), define $g : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ by letting $g(a) = (n + 1) - f(a)$. Then $|g(a) - g(b)| = |f(a) - f(b)| \geqslant |a - b|$, for all $a, b \in \{1, 2, ..., n\}$, and $g(1) = 1$ and $g(n) = n$. Thus, by the first case, we get that $g(a) = a$ and hence $f(a) = n + 1 - a$, for all $a \in \{1, 2, ..., n\}$.

4. Consider the set $\mathcal{P}$ of all points in the plane with integer coordinates. First, two points A, B of $\mathcal{P}$ are painted black. A new point in $\mathcal{P}$ is painted black provided it lies on some circle of rational radius passing through two other black points.

Show that either infinitely many points of $\mathcal{P}$ will be painted black, or else no point of $\mathcal{P}$ other than A and B will be painted black. Furthermore, show that both situations can occur.

Solution. We assume first that at least one other point of $\mathcal{P}$ is painted black. Once this point exists, we argue that the number of black points is infinite. Let R be a back point in $\mathcal{P}$ constructed from a pair of old black points (P, Q) in $\mathcal{P}$. Thus, by assumption the radius of the circle PQR is rational. We produce a new triple (P', Q', R') of black points in $\mathcal{P}$ using the following procedure.

We can form a triangle $P'Q'R'$ so that P, Q, R are midpoints of the sides $Q'R'$, $P'R'$, $P'Q'$ respectively. The construction is the following: we consider lines through P, Q, R parallel to the sides of QR, PR, PQ respectively, and let P', Q', R' denote the intersection points. We claim that:

- the points P', Q', R' have integer coordinates. To see this, consider the parallelogram $PQP'R$, so that the coordinates of P' are computed as

$$P' = Q + R - P$$

which must be integer;

- the radii of the circles through $P'QR$, $Q'PR$, $R'PQ$ are all equal to that of the circle through PQR, since these triangles are all congruent. Since PQR has rational radius, the other three circles must have rational radius as well. As P, Q, R are already painted black, this implies that P', Q', R' are also painted black.

However, the triangle $P'Q'R'$ has area 4-times as big as the area of PQR. Continuing in this fashion we obtain bigger and bigger triangles with black vertices, each containing the previous one, proving that we have infinitely many black points.

If $A(0, 0)$ and $B(2, 0)$ are initially painted black, then a third point painted black will be $C(1, 1)$ since the circle through ABC has radius equal 1. By what we proved above, then this will guarantee that infinitely many points of $\mathcal{P}$ will be painted black.

If $A(0, 0)$ and $B(1, 0)$ are chosen adjacent, we show no other point is painted black. Indeed, let $C(x, y)$ be the first such point so that the radius of the circle through ABC is rational. We will derive a contradiction. Since A, B, C are not collinear, $y \neq 0$. By symmetry, we may assume $x \geq 0$. Otherwise, we can replace C by the point $B'(-1 - x, y)$. Considering the parallelogram $ABCB'$, the circles ACB and ACB' have equal rational radii, and A, C are black. Then B' would then also be painted black and have non-negative x-coordinate.

We compute the radius R of the circle ABC and show it cannot be rational, thus deriving the desired contradiction. First, twice the area of ABC equals

$$AC \cdot BC \cdot \sin \angle C = \text{base } \cdot \text{height} = 2R \sin \angle C \cdot \text{height},$$

using the law of sines in the last equality. Thus

$$AC \cdot BC = 2R \cdot \text{height} = 2R \cdot |y| \in \mathbb{Q}.$$

Since

$$AC^2 = x^2 + y^2, BC^2 = (x-1)^2 + y^2$$

we conclude that

$$(x^2 + y^2)((x-1)^2 + y^2)$$

is the square of a rational number. Since x, y are integers, this expression must be the square of an integer. Let d be the greatest common divisor of x^2+y^2 and $(x-1)^2+y^2$, so that

$$x^2 + y^2 = dU,\ (x-1)^2 + y^2 = dV$$

for some positive integers $(U, V) = 1$. Since the product $(x^2 + y^2)((x-1)^2 + y^2) = d^2UV$ is a perfect square, it follows UV is a perfect square. Since U, V have no common factors, all prime factors of U and V appear with even exponents, or in other words $U = u^2, V = v^2$ for $u, v > 0$. Therefore,

$$x^2 + y^2 = du^2, (x-1)^2 + y^2 = dv^2.$$

Subtracting we find

$$2x - 1 = d(u^2 - v^2).$$

- If $x = 0$, we obtain $d = 1$, $u^2 - v^2 = -1 \implies (u-v)(u+v) = -1 \implies v - u = u + v = 1 \implies u = 0, v = 1 \implies y = 0$, contradicting $y \neq 0$ shown above.
- If $x > 0$, then $2x - 1 > 0$ hence $d(u^2 - v^2) > 0 \implies u > v \implies u \geq v + 1$. This gives

  $$2x - 1 = d(u^2 - v^2) \geq d((v+1)^2 - v^2) = d(2v+1) \geq 2dv + 1 \implies x \geq dv + 1.$$

  But $(x-1)^2 + y^2 = dv^2$ and $y \neq 0$ hence $x < 1 + \sqrt{d}v$. This contradicts $x \geq dv + 1$.

This contradiction shows that only A and B are painted black.