57th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 19, 2014
on the campus of the
University of California, San Diego

PART II: Solutions

1. Alice and Bob each have a bag of 9 balls. The balls in each bag are numbered from 1 to 9. Alice and Bob each remove one ball uniformly at random from their own bag. Let a be the sum of the numbers on the balls remaining in Alice's bag. Let b be the sum of the numbers on the balls remaining in Bob's bag. Determine the probability that a and b differ by a multiple of 4.

Solution. Suppose that Alice removes the ball numbered x from her bag and that Bob removes the ball numbered y from his bag. Then

$$a = 1+2+3+4+5+6+7+8+9-x = 45-x, \quad \text{and}$$
$$b = 1+2+3+4+5+6+7+8+9-y = 45-y.$$

Hence, $a-b = (45-x)-(45-y) = y-x$. Since $1 \le x, y \le 9$, it follows that $-8 \le y-x \le 8$. Hence, in order for $a-b = y-x$ to be a multiple of 4, it must be one of 0, ± 4, or ± 8.

As Alice and Bob each choose 1 ball uniformly randomly from amongst 9, the probability in question is equal to $\frac{n}{9^2}$, where n is the number of pairs (x, y) for which $x-y \in \{0, \pm 4, \pm 8\}$. We proceed to count these now.

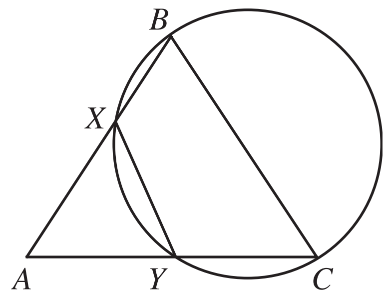
- If $x-y=0$, then $x=y$, meaning we have the pairs $(x,y) = (1,1), (2,2), \ldots, (9,9)$, 9 in total.
- If $x-y=4$, then $x>y$; since $x \le 9$ and $y \ge 1$, the possible pairs are $(x,y) = (5,1), (6,2), (7,3), (8,4), (9,5)$, giving a total of 5. If $x-y=-4$, this is just the same as the above case with the roles of x and y reversed, giving the 5 possibilities $(x,y) = (1,5), (2,6), (3,7), (4,8), (5,9)$.
- If $x-y=8$, there is only one possibility: $(x,y)=(9,1)$. If $x-y=-8$, the only possibility is $(x,y)=(1,9)$.

Hence, $n = 9+5+5+1+1 = 21$, and therefore the probability is $\frac{21}{81} = \boxed{\frac{7}{27}}$.

2. Suppose that $x \in \mathbb{Q}$ is a rational number with the property that $x^2 - x \in \mathbb{Z}$ is an integer. Prove that, in fact, $x \in \mathbb{Z}$ is an integer.

   Solution. Let $a = x^2 - x$; we know that $a \in \mathbb{Z}$. Thus, the rational number x satisfies the quadratic equation $x^2 - x - a = 0$. The general solutions of this equation are $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4a}$. Now, since $a \in \mathbb{Z}$, $1 + 4a \in \mathbb{Z}$ as well; hence, in order for x to be rational, it must be true that $1 + 4a$ is a perfect square: $1 + 4a = k^2$ for some integer k. Since $1 + 4a$ is odd, k is odd. Thus $x = \frac{1}{2}(1 \pm k)$ for an odd integer k; it follows that x is an integer.

3. In triangle ABC, $AB = BC = 25$ and $AC = 30$. The circle with diameter BC intersects AB at X and AC at Y. Determine the length of XY .



Solution. Since BC is a diameter, the inscribed triangle BCY is a right triangle. Since ABC is isosceles, the altitude BY bisects the base, and so $YC = 15$. Hence $BY = \sqrt{25^2 - 15^2} = 20$.

Similarly, BCX is a right triangle. We will determine the length CX as follows: the area of ABC is given both by $\frac{1}{2}(AC)(BY)$ and by $\frac{1}{2}(AB)(CX)$; thus

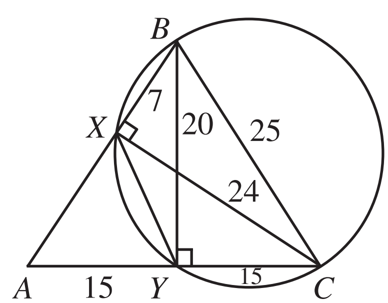
$$(30)(20) = (AC)(BY) = (AB)(CX) = (25)(CX), \quad \therefore\ CX = 24.$$

This allows us to compute BX from Pythagoras again: $BX = \sqrt{25^2 - 24^2} = 7$.

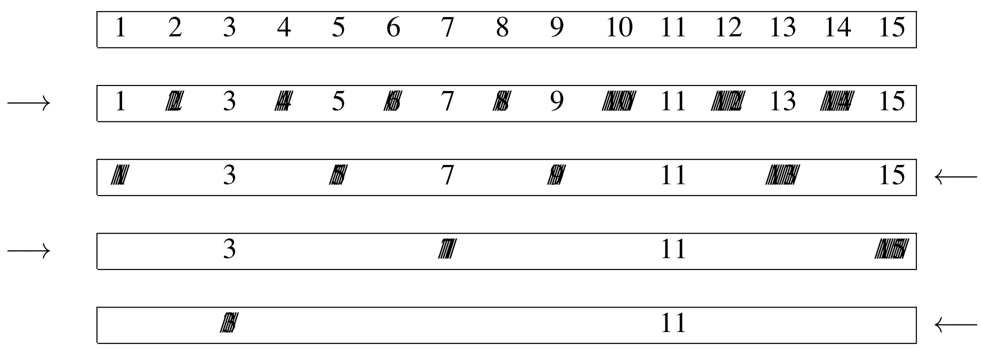
The triangle BXY is *not* a right triangle, but we can use the law of cosines:

$$(XY)^2 = (BX)^2 + (BY)^2 - 2(BX)(BY)\cos\theta = 449 - 280\cos\theta$$

where $\theta = \angle XBY = \angle ABY$. Since ABY is a right triangle, we therefore have $\cos\theta = \frac{20}{25} = \frac{4}{5}$. Hence, we conclude that $(XY)^2 = 449 - 280 \cdot \frac{4}{5} = 225$, and hence $\boxed{\text{XY = 15}}$.



4. A school has a row of n open lockers, numbered 1 through n. Starting at the beginning of the row, you walk past and close every second locker until reaching the end of the row, as shown in the example below. Then you turn around, walk back, and close every second locker that is still open. You continue in this manner back and forth along the row, until only one locker remains open. Define $f(n)$ to be the number of the last open locker. For example, if there are 15 lockers, then $f(15) = 11$ as shown below.



Calculate $f(2014)$.

Solution. To begin, we establish the following recurrence:

$$f(2m) = f(2m-1) = 2m + 1 - 2f(m).$$

To prove this, first consider the case that there are $n = 2m$ lockers. On your first pass, you close lockers $2, 4, 6, \dots, 2m$, so the remaining open lockers are $1, 3, 5, \dots, 2m-1$. On the other hand, if $n = 2m-1$, on the first pass you close lockers $2, 4, 6, \dots, 2m-2$, so the remaining lockers are again $1, 3, 5, \dots, 2m-1$. Thus, everything proceeds exactly the same from here in the two cases, which shows that $f(2m) = f(2m-1)$.

For the second equality, with $n = 2m$ or $2m-1$, after the first pass the remaining open lockers are $1, 3, 5, \dots, 2m-1$; the number of open lockers is m. So from here, you proceed as if there were m lockers, but they've been relabeled: $2m-1 \to 1'$, $2m-3 \to 2'$, …, $1 \to m'$; that is, $p' = 2(m-p)+1$. This means that, if $f(m) = p$, then $f(2m) = p' = 2m + 1 - 2p = 2m + 1 - 2f(m)$, as claimed.

In particular, this means we need only calculate $f(n)$ for odd n; if n is even, $f(n) = f(n-1)$. Now, any odd n is equal to either 1 or 3 mod 4. Iterating the recurrence once, we calculate that

$$\begin{aligned} f(4m+1) = f(2(2m+1)-1) &= 2(2m+1) + 1 - 2f(2m+1) \\ &= 4m + 3 - 2f(2m+2) \\ &= 4m + 3 - 2(2m + 2 + 1 - 2f(m+1)) \\ &= 4f(m+1) - 3. \end{aligned}$$

Similarly

$$\begin{aligned} f(4m+3) = f(2(2m+2)-1) &= 2(2m+2)+1-2f(2m+2) \\ &= 4m+5-2f(2m+2) \\ &= 4m+5-2(2m+2+1-2f(m+1)) \\ &= 4f(m+1)-1. \end{aligned}$$

This allows us to quickly calculate $f(2014)$ iteratively.

$$\begin{gathered} f(2014) = f(2013) = f(4\cdot 503+1) = 4f(504)-3 \\ f(504) = f(503) = f(4\cdot 125+3) = 4f(126)-1 \\ f(126) = f(125) = f(4\cdot 31+1) = 4f(32)-3. \end{gathered}$$

We could do this twice more, or we could now use the original recursion to compute that

$$f(32) = f(2\cdot 16) = 2\cdot 16+1-2f(16) = 33-2f(16) = 33-2f(15),$$

and, as shown in the problem, $f(15) = 11$, so $f(32) = 33-22 = 11$. We now trace back: $f(126) = 4f(32)-3 = 4\cdot 11-3 = 41$; $f(504) = 4f(126)-1 = 4\cdot 41-1 = 163$; $f(2014) = 4f(504)-3 = 4\cdot 163-3 = \boxed{649}$.