The Bergman Space

Let Ω be a bounded domain in $\mathbb{C}^n$ ("domain" = "open" + "connected"). Write $L^2(\Omega)$ for the space of L^2-integrable complex-valued functions on Ω.

That is,

$$L^2(\Omega) = \left\{ f: \Omega \to \mathbb{C} : \int_\Omega |f(z)|^2 dV(z) < \infty \right\}.$$

Here dV is the Euclidean volume element:

$$dV = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \cdots \wedge dx_n \wedge dy_n$$

where $z_j = x_j + i y_j$.

We have the inner product on $L^2(\Omega)$

$$\langle f, g \rangle = \int_\Omega f(z) \overline{g(z)} \, dV(z), \quad f, g \in L^2(\Omega)$$

Def^n: The Bergman space of Ω is defined by $A^2(\Omega) \triangleq L^2(\Omega) \cap H(\Omega)$

(Recall $H(\Omega)$ denotes the space of holomorphic functions on Ω)

We recall $L^2(\Omega)$ is a Hilbert space. we will show $A^2(\Omega)$ is also a Hilbert space. For that, we first prove the following lemma:

In the following, we will write

$$\|f\| = \langle f, f \rangle^{\frac{1}{2}} = \left(\int_\Omega |f(z)|^2 \, dV(z) \right)^{\frac{1}{2}}$$

Lemma 1: Fix $a = (a_1, \cdots, a_n) \in \Omega$ and a polydisc $P(a, r) = \{ z \in \mathbb{C}^n : |z_j - a_j| \le r_j \}$, with $\overline{P(a,r)} \subseteq \Omega$ with $r = (r_1, \cdots, r_n)$.

Then for $\forall f \in A^2(\Omega)$, we have

$$|f(a)| \le \frac{1}{\pi^{\frac{n}{2}} r_1 \cdots r_n} \|f\|$$

Pf: For simplicity of notations, we will assume $a=0$.

Write $\quad f(z) = \sum_{\alpha} C_\alpha z^\alpha$, on $P(a,r)$

$$\Rightarrow \|f\|^2 = \int_\Omega |f(z)|^2 \, dV(z)$$

$$\geq \int_P |f(z)|^2 \, dV(z)$$

$$= \lim_{K\to+\infty} \sum_{|\alpha|,|\beta|\leq K} C_\alpha \overline{C_\beta} \int_P z^\alpha \bar{z}^\beta \, dV(z) = \lim_{K\to\infty} \sum_{|\alpha|\leq K} |C_\alpha|^2 \int_P |z^\alpha|^2 \, dV(z)$$

$$\geq |C_0|^2 \int_P dV = |C_0|^2 \pi^n r_1^2 \cdots r_n^2$$

This establishes Lemma 1.

Corollary 1: Let $K \subseteq \Omega$ be a compact subset.

Then $\exists$ a constant $C = C(K)$, s.t

$\forall f \in A^2(\Omega)$, we have

$$|f(z)| \leq C(K) \|f\|$$

Pf: Fix a compact subset K. Then $\exists$ $r=(r_1,\cdots,r_n)$ with all $r_j>0$, s.t $\overline{P(a,r)}\subseteq\Omega$ for all $a\in K$.

Then the Corollary follows from Lemma 1 if we pick $C(K)=(\pi^{\frac{n}{2}}r_1\cdots r_n)^{-1}$.

Remark: Let $f_n\to f$ in $A^2(\Omega)$. Then by Corollary, $f_n(z)\to f(z)$ uniformly on compact subsets.

Thm 1: $A^2(\Omega)$ is a Hilbert space

Pf: Note $A^2(\Omega)$ is a subspace of $L^2(\Omega)$. Therefore it suffices to show $A^2(\Omega)$ is complete.

Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $A^2(\Omega)$.

Fix any compact subset $K\subseteq\Omega$. By Corollary, we have, $\forall$ k,l

$$|f_k(z)-f_l(z)|\le C(K)\|f_k-f_l\|$$

$\Rightarrow \{f_K(z)\}$ convergences uniformly on K.

$\Rightarrow$ $f_K(z) \to f(z)$ uniformly on every compact K.

for some $f \in H(\Omega)$.

Claim 1: $f \in A^2(\Omega)$

Pf: It suffices to show $f \in L^2(\Omega)$

By Fatou's Lemma

$$\int_\Omega \lim_{k\to\infty} |f_k|^2 dV \le \lim_{k\to\infty} \int_\Omega |f_k|^2 dV < \infty$$

Hence $\|f\| < \infty$

Claim 2: $f_n \to f$ in L^2-norm

Pf: Ex. Hint: By Fatou's Lemma, fixing l,

$$\int_\Omega \lim_{k\to\infty} |f_k - f_l|^2 \le \lim_{k\to\infty} \int_\Omega |f_k - f_l|^2.$$

Recall some defn's from functional analysis.

Defn: Let $\{\varphi_i\}_{i\in I} \subseteq A^2(\Omega)$. We say $\{\varphi_i\}_{i\in I}$ is an orthonormal system if

$$\langle \varphi_k, \varphi_l \rangle = \delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l. \end{cases}$$

Defn:

Let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal system

Let $f \in A^2(\Omega)$. Set

$$a_k = \langle f, \varphi_k \rangle = \int_\Omega f(z)\, \overline{\varphi_k(z)}\, dV(z) \in \mathbb{C}$$

We call a_k the Fourier coefficient of f w.r.t $\{\varphi_k\}$. We call $\sum_{k=1}^{\infty} a_k \varphi_k$ the Fourier Series of f w.r.t $\{\varphi_k\}$.

Proposition 1 (Bessel inequality)

Let $\{\varphi_k\}$ be an orthonormal system of $A^2(\Omega)$. Let $\{a_k\}$ be the Fourier Series of f w.r.t to $\{\varphi_k\}$. Then $\sum_{k=1}^{\infty} |a_k|^2 \leq \|f\|^2$.

Consequently, $\sum_{k=1}^{\infty} a_k \varphi_k$ is a Cauchy sequence

Pf: $\forall N \geq 1$,

$$0 \leq \|f - \sum_{k=1}^{N} a_k \varphi_k\|^2 = \int_\Omega (f - \sum_{k=1}^{N} a_k \varphi_k)(\bar{f} - \sum_{k=1}^{N} \bar{a}_k \bar{\varphi}_k)\, dV(z)$$

$$= \int |f|^2 dV - \int_\Omega \sum_{k=1}^{N} a_k \varphi_k \overline{\sum_{k=1}^{N} a_k \varphi_k}\, dV(z)$$

$$= \|f\|^2 - \sum_{k=1}^{N} |a_k|^2$$

<u>Proposition 2</u>: Let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal system of $A^2(\Omega)$. Fix any compact subset K.
$\exists\, C = C(K)$ s.t $\forall z \in K$, $\forall N \in \mathbb{Z}^+$, we have

$$\sum_{k=1}^{N} |\varphi_k(z)|^2 \le C$$

Pf: Fix any $z \in K$, set

$$f(\zeta) = \sum_{k=1}^{N} \overline{\varphi_k(z)}\, \varphi_k(\zeta).$$

Since $\{\varphi_k\}_{k=1}^{\infty}$ is o.n.s $\Rightarrow$ $\|f\|^2 = \sum_{k=1}^{N} |\varphi_k(z)|^2$

By Corollary 1 in the above, $\exists\, C_1 = C_1(K)$
s.t $|f(z)| \le C_1 \|f\|$ $\Rightarrow$

$$|f(z)|^2 \le C_1^2 \|f\|^2$$

$\Rightarrow$

$$\left(\sum_{K=1}^{N} |\varphi_K(z)|^2\right)^2 \le C_1^2 \sum_{K=1}^{N} |\varphi_K(z)|^2$$

$$\Rightarrow \quad \sum_{K=1}^{N} |\varphi_K(z)|^2 \le C_1^2$$

This proves proposition 2 if we choose $C = C_1^2$.

proposition 3: Let $\{\varphi_K\}_{K=1}^{\infty}$ be an o.n.s in $A^2(\Omega)$. Assume $\sum_{K=1}^{\infty} |a_K|^2 < \infty$.

Then we have

(i) $\sum_{K=1}^{\infty} a_K \varphi_K$ converges uniformly on compact subsets to some $g \in A^2(\Omega)$

(ii) $\lim_{N \to \infty} \| g - \sum_{K=1}^{N} a_K \varphi_K \| = 0$

(iii) a_k is the Fourier coefficient of g, i.e., $a_k = \langle g, \varphi_k \rangle$.

Pf: (i) (ii): Ex.

Hint: $\left\{ \sum_{k=1}^{N} a_k \varphi_k \right\}_{N=1}^{\infty}$ is a Cauchy sequence in $A^2(\Omega)$.

(iii): By (ii) $\| g - \sum_{k=1}^{N} a_k \varphi_k \| \to 0$

By Schwarz inequality for the Hilbert space $A^2(\Omega) \Rightarrow$

$$|\langle g - \sum_{k=1}^{N} a_k \varphi_k, \varphi_k \rangle| \leq \| g - \sum_{k=1}^{N} a_k \varphi_k \| \, \| \varphi_k \|$$

Hence $\lim_{N\to\infty} \langle g - \sum_{k=1}^{N} a_k \varphi_k, \varphi_k \rangle = 0$

$\Rightarrow \langle g, \varphi_k \rangle = a_k$

Thm. Let $\{\varphi_k\}_{k=1}^{\infty}$ be an o.n.s in $A^2(\Omega)$.

Set $K(z,\bar{\xi}) = \sum_{k=1}^{\infty} \varphi_k(z)\overline{\varphi_k(\xi)}$.

Then K is holomorphic in $z \in \Omega$, when ξ fixed

K is holomorphic in $\bar{\xi} \in \bar{\Omega} = \{\bar{\xi} : \xi \in \Omega\}$.

when z fixed

(or K is anti-holomorphic in $\xi \in \Omega$)

$\Rightarrow$ K is holomorphic in $(z,\bar{\xi}) \in \Omega \times \bar{\Omega}$.

Pf: By proposition 2,

$$\sum_{k=1}^{\infty} |\varphi_k(z)|^2, \quad \sum_{k=1}^{\infty} |\varphi_k(\xi)|^2 < \infty$$

for fixed $z, \xi \in \Omega$, respectively

Then the first 'seperately holomorphic' part follows from proposition 3.

The last 'jointly holomorphic' part follows from Hartogs' thm.

Remark: If $\{\varphi_k\}_{k=1}^{\infty}$ is an o.n.b (i.e. it is an o.n.s and is also complete), then $K(z,\bar{\zeta})$ is called the Bergman kernel of Ω.

Q1: Do countable o.n.b exist?

Q2: Does the Bergman kernel K depend on the choice of the o.n.b?