Levi pseudoconvex

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Defn: Let G be an open set in $\mathbb{R}^N$ with $p \in \partial G$.
We say G has C^k ($k \le \infty$) smooth boundary at $p \in \partial G$ if $\exists$ a small nbhd $U \subset \mathbb{R}^N$ and a real-valued function $\varphi \in C^k(U)$ s.t

(i) $U \cap G = \{x \in U : \varphi(x) < 0\}$

(ii) $d\varphi(x) \neq 0$, $x \in U$

If ∂G has C^k smooth boundary at $\forall p \in \partial G$, then we say G has C^k-smooth boundary.
Any function φ satisfying (i), (ii) is called a local defining function at p

proposition: If G is a bounded domain in $\mathbb{R}^N$ with C^k boundary, then $\exists$ a global defining function of G. That is, $\exists$ a nbhd U of ∂G and $\varphi \in C^k(U)$ s.t

$$G \cap U = \{x \in U : \varphi(x) < 0\}.$$

Defn: Let $G \subset \mathbb{R}^N$ with C^1 boundary. φ is a local defining function at $p \in \partial G$.
set

$$T_p(\partial G) = \{\xi \in \mathbb{R}^N : \sum_{j=1}^{N} \frac{\partial \varphi}{\partial x_j}(p)\xi_j = 0\}.$$

as the tangent space of ∂G at p.

Thm: Let G be a domain in $\mathbb{R}^N$ with C^2 smooth boundary. Then TFAE:

(1) G is convex

(2) G is geometrically convex

(3) For every $K \subset\subset G$,

$$\hat{K} = \{z \in \mathbb{R}^N : l(z) \le \sup_{\xi \in K} l(\xi) \text{ for } \forall \text{ real linear } L\}.$$

$\subset\subset G$

(4) For every $p \in \partial G$ and every local defining function φ of ∂G at p. it holds that

$$\sum_{j,k=1}^{N} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(p)\xi_j \xi_k \ge 0$$

for $\forall \xi \in T_p(\partial G)$.

Remark: • If it holds that

$$\sum_{j,k=1}^{N} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(p)\xi_j \xi_k > 0, \ \xi \in T_p(\partial G), \xi \neq 0. \text{ Then}$$

we say G is strictly convex at p.

• If G is strictly convex at every $p \in \partial G$, we say G is strictly convex.

Now we discuss the pseudoconvexity setting.

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Definition: Let $G \subset \mathbb{C}^n$ be a domain with C^1 boundary with local defining function φ at p. The holomorphic tangent space of ∂G at p is defined as:

$$T_p^{(1,0)}(\partial G) = \{ w \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j}(p) w_j = 0 \}.$$

Definition: We define the Levi form of ∂G at p.

$$L_p(\varphi, w) = \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(p) w_j \overline{w_k}$$

where $w \in T_p^{(1,0)}(\partial G)$.

We say ∂G is Levi pseudoconvex at p if $L_p(\varphi, w) \geq 0$ for $\forall\, w \in T_p^{(1,0)}(\partial G)$.

If ∂G is Levi pseudoconvex at every $p \in \partial G$, we say ∂G is Levi pseudoconvex.

In particular, if $L_p(\varphi, w) > 0$ for $\forall\, 0 \neq w \in T_p^{(1,0)}(\partial G)$, we say ∂G is strictly Levi pseudoconvex at p.

Thm: Let $G \subset \mathbb{C}^n$ be a domain with C^2 boundary. Assume G is convex, $\Rightarrow$ G is Levi pseudoconvex.

Pf: One can verify directly by definition. It also follows from the following theorem.

Thm: Let $G \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary. Then G is pseudoconvex iff G is Levi pseudoconvex.

Pf. "$\Rightarrow$" Assume G is pseudoconvex. We have proved that $-\log d(z, \partial G) \in PSH(G)$. As G has C^2 boundary, $d(z) = d(z, \partial G)$ is a C^2 function in U for some nbhd U of ∂G. Then

$$L_z(-\log d, w) \geq 0, \quad z \in U, \ w \in \mathbb{C}^n.$$

Pick a defining function r of G as

$$r(z) = \begin{cases} -d(z) & z \in G \\ 0 & z \in \partial G \\ d(z) & z \in \mathbb{C}^n - G. \end{cases}$$

Again $r(z) \in C^2(U)$. We compute. When $z \in G$.

$$\frac{\partial^2(-\log d(z))}{\partial z_j \partial \bar{z}_k} = -\frac{1}{d(z)} \frac{\partial^2 d(z)}{\partial z_j \partial \bar{z}_k} + \frac{1}{d^2(z)} \frac{\partial d(z)}{\partial z_j} \frac{\partial d(z)}{\partial \bar{z}_k}$$

$$= -\frac{1}{r(z)} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} + \frac{1}{r^2(z)} \frac{\partial r(z)}{\partial z_j} \frac{\partial r(z)}{\partial \bar{z}_k}$$

Thus

$$L_z(-\log d, w) = -\frac{1}{r(z)} L_z(r, w) + \frac{1}{r^2(z)} \left| \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j} w_j \right|^2$$

$$L_z(-\log d, w) = -\frac{1}{r(z)} L_z(r,w) + \frac{1}{r^2(z)} \left| \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j} w_j \right|^2$$

For a fixed z and $w \in \mathbb{C}^n$ satisfying $\sum_{j=1}^{n} \frac{\partial^2 r(z)}{\partial z_j} w_j = 0$.

we have since $L_z(-\log d . w) \geq 0 \Rightarrow$

$$L_z(r,w) \geq 0 \text{ for } w \in \mathbb{C}^n \text{ satisfying } \sum_{j=1}^{n} \frac{\partial^2 r(z)}{\partial z_j} w_j = 0.$$

Let $z \to p \in \partial G$. we get

$$L_p(r,w) \geq 0 \text{ for } w \in T_p^{(1,0)}(\partial G).$$

Hence ∂G is Levi pseudoconvex at p. for $\forall p \in \partial G$

" $\Leftarrow$ " Assume ∂G is Levi pseudoconvex at $\forall p \in \partial G$. Let ρ be a local defining function at p.

<u>Claim</u>: Levi pseudoconvexity at $\forall p \in \partial G$

$\Leftrightarrow$ For every $p \in \partial G$, $\exists$ a nbhd U of p and $C > 0$ s.t

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_k} w_j \overline{w_k} + C \|w\| \left| \sum_{j=1}^{n} \frac{\partial \rho(z)}{\partial z_j} w_j \right| \geq 0 \quad (*)$$

for $\forall z \in \partial G \cap U$, $\forall w \in \mathbb{C}^n$.

we only prove " $\Rightarrow$ ":

Pf: At every $z \in \partial G$, write $\vec{n}(z) = \left(\frac{\partial \rho}{\partial z_1}, \cdots, \frac{\partial \rho}{\partial z_n} \right)\Big|_z$

Then $w \in T_z^{1,0}(\partial G) \Leftrightarrow \langle w, \vec{n}(z) \rangle = 0$

We decompose $w \in \mathbb{C}^n$ into $w = w' + w''$.

where $\langle w', \vec{n}(z) \rangle = 0$, $\langle w'', \vec{n}(z) \rangle = \langle w, \vec{n}(z) \rangle$.

Then
$$\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} w_j \overline{w_k}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} w_j' \overline{w_k'} + O(\|w\|\|w''\|)$$

$$\geq A\|w\|\|w''\|, \quad A \in \mathbb{R} \text{ depends on } \left|\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right| \text{ on } U$$

$$\geq -C\|w\| \left|\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j\right|$$

This is because

$$\|w''\| = \frac{\|\langle w'', \vec{n}(z)\rangle\|}{\|\vec{n}(z)\|}$$, and C depends on $\|\vec{n}(z)\|$ on U ▨

By shrinking U if necessary, and apply Implicit function theorem we can assume ρ takes the form

$$\phi(\mathrm{Re}\, z_1, \mathrm{Im}\, z_1, \cdots, \mathrm{Re}\, z_{n-1}, \mathrm{Im}\, z_{n-1}, \mathrm{Re}\, z_n) = \mathrm{Im}\, z_n$$

Then in this case, since ρ depends lineary on $\mathrm{Im}\, z_n$, all derivatives of ρ all independent of $\mathrm{Im}\, z_n$. We have $(*)$ still hold for $z \in U \cap G$.

Write $u = -\log(-\rho)$ in U, then in $U \cap G$

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} w_j \overline{w_k}$$

$$= \frac{1}{|\rho|} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} w_j \overline{w_k} + \frac{1}{\rho^2} \sum_{j,k=1}^{n} \frac{\partial \rho}{\partial z_j} \frac{\partial \rho}{\partial \bar{z}_k} w_j \overline{w_k}$$

$$\geq \frac{-2\left(\frac{C}{2}\right)}{|\rho|} \|w\| \left|\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} w_j\right| + \frac{1}{\rho^2} \left|\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j\right|$$

$$\geq -\frac{C^2}{\cdot}\|w\|^2 - \frac{1}{\rho^2}\left|\sum^{n} \frac{\partial \rho}{\partial z_j}(z) w_j\right| + \frac{1}{\rho^2}\left|\sum \frac{\partial \rho(z)}{\partial z_j} w_j\right|$$

$$\geq -\frac{c^2}{4}\|W\|^2 - \frac{1}{\rho^2}\left|\sum_{j=1}^{n}\frac{\partial \rho}{\partial z_j}(z)w_j\right| + \frac{1}{\rho^2}\left|\sum_{j=1}^{n}\frac{\partial \rho(z)}{\partial z_j}w_j\right|$$

(As $-2ab \geq -a^2-b^2$)

$\Rightarrow$ $u(z) + \frac{c^2}{4}\|z\|^2$ is psH in $G \cap U$.

Assume $U = B(p, r)$ for some r Then the function $\max\{-\log(r - \|z-a\|), u(z) + \frac{c^2}{4}\|z\|^2\}$ is a psH exhaustion function for $B \cap G$. Thus $B \cap G$ is pseudoconvex. It follows from the next thm that G is pseudoconvex.

Thm: Let Ω be an open set in $\mathbb{C}^n$. If to every pt in $\partial\Omega$ there is a nbhd W such that $W \cap \Omega$ is pseudoconvex, then Ω is pseudoconvex.

Pf: See P47 in Hörmander's book.