Levi problem

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In this section, we will give a solution to the Levi problem assuming the solvability of $\bar\partial$-eqn.

Thm: Let Ω be a domain in $\mathbb{C}^n$ such that the eqn $\bar\partial u = f$ has a solution $u \in C^\infty_{(0,q)}(\Omega)$ for every $f \in C^\infty_{(0,q+1)}(\Omega)$ with $\bar\partial f = 0$. (Here $q = 0, 1, \cdots, n-2$). Then Ω is a domain of holomorphy.

Pf. We will prove by induction on n.

Step 1: If $n=1$, the theorem is true since every domain in $\mathbb{C}$ is a domain of holomophy.

Step 2. Assume the statement holds for $n-1$. Now consider $\Omega \subset \mathbb{C}^n$.

Fix $p_0 \in \partial\Omega$ and a change of coordinates s.t $p_0 = 0$ under the new coordinates and $p_0 \in \partial\tilde\Omega$, where

$$\tilde\Omega = \Omega \cap \{z \in \mathbb{C}^n : z_n = 0\} = \{z \in \Omega : z_n = 0\}.$$

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If we can prove the solvability of $\bar{\partial}$ eqn on $\tilde{\Omega}$. Then by inductive hypothesis, $\tilde{\Omega}$ is a domain of holomorphy. Then $\exists$ a $f \in H(\tilde{\Omega})$ s.t f cannot be holomorphically extended across P_0. Then we prove $\exists F \in H(\Omega)$ s.t $F|_{\tilde{\Omega}} = f$. Thus F cannot be holomorphically extended across P_0. As P_0 is arbitrary, we conclude Ω is a domain of holomorphy.

For that, we set the projection,

$$\pi : (z_1, \cdots, z_n) \to (z_1, \cdots, z_{n-1}, 0)$$

Define $G = \{ z \in \Omega : \pi(z) \notin \tilde{\Omega} \}$

Lemma 1: G is closed in Ω.

Pf: Note $\Omega - G = \{ z \in \Omega : \pi(z) \in \tilde{\Omega} \}$.

As $\tilde{\Omega}$ is open, $\Rightarrow \Omega - G$ is open $\Rightarrow G$ is closed in Ω.

Lemma 2: $\tilde{\Omega}$ is closed in Ω.

Pf: easy.

By Lemma 1,2. $\exists$ a smooth function h on Ω.

s.t $h \equiv 1$ in a nbhd of $\tilde{\Omega}$ and $h \equiv 0$ in a nbhd of G.

<u>Claim</u>: For every smooth form $w \in C^\infty_{(0,q+1)}(\tilde{\Omega})$ with $\bar\partial w = 0$, there $\exists$ $\phi \in C^\infty_{(0,q)}(\tilde{\Omega})$ s.t $\bar\partial \phi = w$.

Pf: First. let $\tilde{w} = \pi^*(w)$ i.e.

$$\tilde{w}(z_1, \cdots, z_n) = w(z_1, \cdots, z_{n-1}, 0).$$ This is defined on $\Omega - G$. Then $h \cdot \tilde{w}$ is a $(0,q+1)$-form on Ω.

Similarly, $(z_n)^{-1} \bar\partial h \wedge \tilde{w}$ is a well-defined $(0,q+2)$ form. Moreover, it satisfies

$$\bar\partial \left[(z_n)^{-1} \bar\partial h \wedge \tilde{w} \right] = 0.$$

By the hypothesis, $\exists$ a $v \in C^\infty_{(p,q)}(\Omega)$ s.t

$$\bar\partial v = (z_n)^{-1} \bar\partial h \wedge \tilde{w}$$

Now let $F = h \cdot \tilde{w} - z^n \cdot v$ then $\bar\partial F = 0$. Thus

$\exists\, H \in C^{\infty}_{(0,q-1)}(\Omega)$ s.t $\bar\partial H = F$.

It follows that $(\bar\partial H)|_{\tilde\Omega} = F|_{\tilde\Omega} = \tilde w|_{\tilde\Omega}$

$\Longrightarrow \quad \bar\partial (H|_{\tilde\Omega}) = w$. This implies $\bar\partial \phi = w$ has a solution.

<u>claim</u>: Let $f \in H(\tilde\Omega)$. Then $\exists\, F \in H(\Omega)$ s.t

$$F|_{\tilde\Omega} = f.$$

Pf: Let $\tilde f = \pi^* f$. Note, similarly as above, $(z_n)^{-1}\bar\partial h \cdot \tilde f$ is well-defined and C^∞ in Ω. Moreover,

$$\bar\partial (z_n^{-1} \bar\partial h \cdot \tilde f) = 0$$

Thus $\exists\, g \in C^\infty(\Omega)$. s.t $\bar\partial g = (z_n)^{-1} \bar\partial h \cdot f$.

Set $F = h\cdot f - z_n g$. Then $\bar\partial F = 0$, thus $F \in H(\Omega)$

It is easy to see $F|_{\tilde\Omega} = f$.

A related well-known result:

Thm: (Ohsawa–Takegoshi) Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $\psi : \Omega \to [-\infty, +\infty)$ a plurisubharmonic function and $H \subset \mathbb{C}^n$ a complex hyperplane. Then $\exists$ a constant C depending only on the diameter of Ω such that: for any holomorphic function f on $\Omega \cap H$ satisfying any holomorphic function f on $\Omega\cap H$ satisfying

$$\int_{\Omega\cap H} e^{-\psi}|f|^2 dV_{n-1} < \infty$$

where dV_{n-1} denotes the $(2n-2)$-dimensional Lebesgue measure, there exists $F \in H(\Omega)$ s.t $F|_{\Omega\cap H} = f$ and

$$\int_{\Omega} e^{-\psi}|F|^2 dV_n \leq \int_{\Omega\cap H} |f|^2 e^{-\psi} dV_{n-1}$$