Recall: Let Ω be a domain in $\mathbb{C}^n$.

Defn: Ω is pseudoconvex if it has a continuous plurisubharmonic function.

Defn: Let $K \subset\subset \Omega$. We define the plurisubharmonic hull of K (psh-hull of K)

$$\hat{K}^P_\Omega = \{z \in \Omega : u(z) \le \sup_{\xi \in K} u(\xi) \text{ for every } u \in Psh(\Omega)\}$$

Remark 1: Let $z_0 \in \Omega$, $a \in \mathbb{C}^n$. Write $D = \{z_0 + \lambda a : \lambda \in \mathbb{C}, |\lambda| < r\}$ for some $r > 0$. Assume $\bar{D} \subset \Omega$. Let K be subset of Ω containing ∂D. Then $\bar{D} \subset \hat{K}^P_\Omega$. Indeed, let $u \in Psh(\Omega)$. Then $u(z_0 + \lambda a)$ is subharmonic in $\{|\lambda| \le r\}$. By the maximum principle, $u(z_0 + \lambda a) \le \sup_{|\lambda| = r} u(z_0 + \lambda a) \le \sup_{\xi \in K} u(\xi)$

Remark 2: Recall the holomorphic hull of K:

$$\hat{K}_\Omega = \{z \in \Omega : |f(z)| \le \sup_{\xi \in K} |f(\xi)| \text{ for every } f \in H(\Omega)\}$$

It is easy to see $\hat{K}^P_\Omega \subset \hat{K}_\Omega$.

Proposition: If Ω is a domain of holomorphy (holomorphically convex), then for every $K \subset\subset \Omega$, $\hat{K}^P_\Omega \subset \hat{K}_\Omega \subset\subset \Omega$.

Thm: Let Ω be a domain in $\mathbb{C}^n$. Then TFAE:

(1) $-\log d(z, \Omega^c)$ is plurisubharmonic in Ω.

(2) Ω is pseudoconvex

(3) For every $K \subset\subset \Omega$, $\hat{K}^P_\Omega \subset\subset \Omega$.

proof: "(1) $\Rightarrow$ (2)" Let $u(z) = \sum_{i=1}^n |z_i|^2 - \log d(z, \Omega^c)$. Then u is a continuous plurisubharmonic exhaustion function of Ω. Thus (2) holds.

"(2) $\Rightarrow$ (3)". Let $u(z)$ be a continuous plurisubharmonic exhaustion function of Ω. Then

$$\hat{K}^P_\Omega \subset \{z \in \Omega : u(z) \le \sup_{\xi \in K} u(\xi)\}.$$

Hence $\hat{K}^P_\Omega \subset\subset \Omega$.

"(3) $\Rightarrow$ (1)" . Assume $\hat{K}^P_\Omega \subset\subset \Omega$ for every $K \subset\subset \Omega$.

Hence $K_\Omega' \subset\subset \Omega$.

"(3)⇒(1)" . Assume $\hat{K}^P_\Omega \subset\subset \Omega$ for every $K \subset\subset \Omega$.

we need to prove $-\log d(z, \Omega^c)$ is a plurisubharmonic in Ω.

Let $z_0 \in \Omega$, $w \in \mathbb{C}^n$. choose $r>0$ so that

$$D := \{z_0 + \lambda w : |\lambda| \le r\}$$

Let $p(z)$ be a holomorphic polynomial such that

$$-\log d(z_0+\lambda w, \Omega^c) \le \operatorname{Re} p(\lambda), \quad |\lambda| = r$$

That is,

$$d(z_0+\lambda w, \Omega^c) \ge |e^{-p(\lambda)}|, \quad |\lambda| = r \qquad (*)$$

we need to show the same estimate holds for $|\lambda| \le r$.

To do so, we first pick any vector $v \in \mathbb{C}^n$ with $\|v\| < 1$. Consider for $0 \le t \le 1$ the mapping

$$\lambda \to z_0 + \lambda w + t v e^{-p(\lambda)}, \quad |\lambda| \le r$$

We denote the image by D_t. i.e.

$$D_t = \{z_0 + \lambda w + t v e^{-p(\lambda)} : |\lambda| \le r\}.$$

Note $D = D_0 \subset \Omega$. Set

$$\Lambda = \{0 \le t \le 1 : D_t \subset \Omega\}$$

Clearly $\Lambda \ne \phi$ as $0 \in \Lambda$.

Note each D_t is compact. Then Λ is open in $[0,1]$.

we will prove Λ is also closed in $[0,1]$.

Let $K = \{z_0 + \lambda w + t v e^{-p(\lambda)} : |\lambda| = r, \ 0 \le t \le 1\}$

By $(*)$, K is compact in Ω.

If $t \in \Lambda$, by Remark 1,

$$D_t \subset \hat{K}^P_\Omega \quad (\text{for } t \in \Lambda)$$

By (iii), $\hat{K}^P_\Omega \subset\subset \Omega$. i.e $\overline{\hat{K}^P_\Omega}$ is compact in Ω.

Then $\overline{\bigcup_{t\in\Lambda} D_t} \subset \Omega \Rightarrow \Lambda$ is closed in $[0,1]$

Thus $\Lambda \ne \phi$ and is closed and open in $[0,1]$

$\Rightarrow \quad \Lambda = [0,1]$.

That is $z_0 + \lambda w + t v e^{-p(\lambda)} \in \Omega$.

It holds for all $|\lambda| \le r$, $0 \le t \le 1$, $\|v\| < 1$

Hence $d(z_0+\lambda w, \Omega^c) \ge |e^{-p(\lambda)}|$. for all $|\lambda| \le r$

$\Rightarrow \quad -\log d(z_0+\lambda w, \Omega^c) \le \operatorname{Re} p(\lambda)$ for all $|\lambda| \le r$.

Hence $d(z_0+\lambda w, \Omega^{c}) \geq |e \quad |$ for all $|\lambda| \leq r$

$$\Rightarrow \quad -\log d(z_0+\lambda w, \Omega^c) \leq \operatorname{Re} p(\lambda) \text{ for all } |\lambda| \leq r.$$

Corollary: If Ω is a domain of holomorphy, then Ω is pseudoconvex.

Proof: Ω is a domain of holomorphy $\Rightarrow$ Ω is holomorphically convex.

Let $K \subset\subset \Omega$. Then $\hat{K}_\Omega \subset\subset \Omega$.

By Remark 2, as $\hat{K}^P_\Omega \subset \hat{K}_\Omega \subset\subset \Omega$.

Thus (iii) in Thm is satisfied. $\Rightarrow$

Ω is pseudoconvex.

A natural question is the converse direction: Is every pseudoconvex domain a domain of holomorphy? This is the well-known Levi problem (raised by Levi in 1910). The problem was solved in the affirmative independently by Oka (1953) Norguent (1954), Bremermann (1954). The general solution of Levi problem on complex manifold was given by Grauert (1958). In 1960's, Kohn, Hörmander gave a solution to Levi problem using L^2-estimate of $\bar{\partial}$-equation. We will in the next chapter use this approach to give a solution to Levi-problem.

Here we desripe the Levi-problem on manifold.

Defn: Let M be a complex manifold. We say M is a stein manifold if it satisfies the following three conditions:

(1) M is holomorphically convex. That is, for every $K \subset\subset M$;

$$\hat{K}_M = \{x \in M : |f(x)| \leq \sup_{x \in K} |f|, \text{ for any } f \in H(M)\}$$

(2). For every $x \neq y$ on M, $\exists f \in H(M)$, such that

$$f(x) \neq f(y)$$

One can show that every stein manifold is pseudoconvex, i.e, there is a continuous plurisubharmonic function of M.

Levi problem on stein manifolds asks: Let M be a complex manifold. Assume M is pseudoconvex. Is M a stein manifold?

The importance of the problem is: In general, it is difficult to find holomorphic functions on a complex manifold M. But it is much easier to find plurisubharmonic function on M. If the Levi problem is answered affirmatively, then once we know M is pseudoconvex, there must be plenty of holomorphic functions on M (to separate points).