Plurisubharmonic functions

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Defn: (Subharmonic function)

Let Ω be a domain in $\mathbb{C}$, u is a real-valued function on Ω:

$$u: \Omega \to \mathbb{R} \cup \{-\infty\}. \quad (u \not\equiv -\infty)$$

We say u is subharmonic if

(i) u is upper semi-continuous

(ii) For any closed disc $\overline{\Delta(z_0, r)}$ $\left(= \{z \in \mathbb{C} : |z - z_0| \leq r\}\right)$ in Ω, it always holds that

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta.$$

Remark: upper semi-continuous at a means

$$\varlimsup_{\substack{z \to a \\ z \neq a}} u(z) \leq u(a)$$

Proposition: Let u be a subharmonic function in Ω, and let φ be an increasing convex function on $\mathbb{R}$,

Then $\varphi(u)$ is also subharmonic in Ω.

Proof: If u is subharmonic, then

(i) u is upper semicontinuous, as φ is $\uparrow$,

we have $\varphi(u)$ is upper semicontinuous.

(ii) For any $\overline{\Delta(a, r)} \subset \Omega$. we have,

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta$$

Since φ is increasing and convex, by Jensen Inequality

$$\varphi(u(a)) \leq \varphi\left(\frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta\right)$$
$$\leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(u(a + re^{i\theta}))\, d\theta$$

It is called 'subharmonic' in the sense of the next theorem

Theorem: Let u be an upper semi-continuous

$$u: \Omega \to \{-\infty\} \cup \mathbb{R}.$$

Then TFAE:

(i) u is subharmonic

(ii) For every $G \subset\subset \Omega$ and any h that is continuous in $\overline{G}$, and harmonic in G, if $u(z) \leq h(z)$ on ∂G, then $u(z) \leq h(z)$ in G.

A useful way to check subharmonicity:

Lemma0: Let u be an upper semicontinuous function in Ω. u is subharmonic in Ω if and only if for every closed disc $\overline{D} \subset \Omega$, and every holomorphic polynomial $p(z)$ with $u \leq \operatorname{Re} p(z)$ on ∂D it is always true that $u \leq \operatorname{Re} p(z)$ in D

proof: "$\Rightarrow$" Assume u is subharmonic. Let $p(z)$ be a holomorphic polynomial and $u \leq h$ on ∂D, where $h = \operatorname{Re} p(z)$. Note h is harmonic.

Set $u_1 = u - h$. Then u is subharmonic and upper semicontinuous, By assumption, $u_1 \leq 0$ on ∂D. We need to show $u_1 \leq 0$ on D. Suppose $u_1 > 0$ at some $z_0 \in D$. Since u_1 is upper semicontinuous, $\sup_{\xi \in \overline{D}} u_1(\xi)$ is attained at some $a \in \overline{D}$: (why?)

$$u_1(a) = \sup_{\xi \in \overline{D}} u_1(\xi) \geq u_1(z_0) > 0$$

Moreover, as $u_1 \leq 0$ on $\partial D \Rightarrow a \in D$.

Let $E = \{z \in \overline{D} : u(z) = \sup_{\xi \in \overline{D}} u_1(\xi)\}$. Then

$E \neq \phi$ and $E \subset\subset D$.

(Indeed, E is compact. why?)

Pick $b \in \partial E$. Then $\exists\, r > 0$ s.t $\overline{D(b,r)} \subset D$.

Note $\partial D(b,r)$ must have an open piece of arc NOT contained in E. Then

$$u_1(b) = \sup_{\xi \in \overline{D}} f(\xi) > \frac{1}{2\pi} \int_0^{2\pi} u_1(b + re^{i\theta})\, d\theta$$

But this contradicts with the fact that u_1 is subharmonic.

"$\Leftarrow$" Assume for every $\overline{D} \subset \Omega$ and every holomorphic polynomial $p(z)$ with $u \le h := \operatorname{Re} P(z)$ on ∂D, it holds that $u \le h$ on D. Fix $D = D(z_0, r)$, we will show .

$$u(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$$

The upper semicontinuity of u implies there is a strictly decreasing sequence of continuous functions u_n on ∂D (with a uniform upper bound) such that

$$\lim_{n\to\infty} u_n = u .$$

Since Trig functions are dense in $C^0(\partial D, \mathbb{R})$. $\Rightarrow$

$$\exists\ P_n(z) \text{ such that } \quad u_n \le \operatorname{Re} P_n(z) \le u_n + \frac{1}{n}$$

By assumption,

$$u_n(z_0) \le \operatorname{Re} P_n(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} P_n(z_0 + re^{i\theta})\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\left(u_n + \frac{1}{n}\right) d\theta$$

We use monotone convergence Thm and let $n \to \infty$.

Corollary: (Maximum principle for subharmonic function)

Let u be subharmonic in Ω and $G \subset\subset \Omega$. Then

$$u(z) \le \sup_{\xi \in \partial G} u(\xi) \quad \text{for all } z \in G.$$

Thm: Let u be in $C^2(\Omega)$. Then u is subharmonic in Ω if and only if $\Delta u \ge 0$ in Ω.

Proposition: Let $f \in H(\Omega)$ and $f \not\equiv 0$. Then

$\log|f|$ and $|f|^p$, $0 < p < \infty$ are all subharmonic.

Proof: We first use the lemma above to show it for $\log|f|$.

(1) Clealy $\log|f|$ is upper Semi-continuous

Fix $\overline{D} \subset \Omega$.

Let $p(z)$ be a holomorphic polynomial such that

$$\log|f| \leq \operatorname{Re} p(z) \quad \text{on } \partial D. \text{ That is}$$

$$|f| \leq e^{\operatorname{Re} p(z)} = |e^{p(z)}| \quad \text{or}$$

$$\left|\frac{f}{e^{p(z)}}\right| \leq 1 \qquad \text{on } \partial D$$

By maximum principle, $\left|\frac{f}{e^{p(z)}}\right| \leq 1$ on D.

$$\Rightarrow \quad |f| \leq |e^{p(z)}| = e^{\operatorname{Re} p(z)} \quad \text{on } D$$

$$\Rightarrow \quad \log|f| \leq \operatorname{Re} p(z) \quad \text{on } D.$$

By the lemma, $\log|f|$ is subharmonic.

(2) To show $|f|^p$, $0<p<\infty$ is subharmonic we note e^{px} is increasing and convex. Then it follows from (1) and the proposition that $|f|^p$ is subharmonic.

Part II: Plurisubharmonic function.

Defn: A function $u: \Omega \to [-\infty, +\infty)$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to plurisubharmonic if

(a) u is upper semicontinuous.

(b) For every complex line $L \subset \mathbb{C}^n$, $u|_{\Omega \cap L}$ is subharmonic on $\Omega \cap L$.

That is, for every $z_0 \in G$ and $a \in \mathbb{C}^n$, $u(z_0 + \lambda a)$, is subharmonic in $\lambda \in \mathbb{C}$ where it is defined.

Write $Psh(\Omega)$ be the set of all functions that is plurisubharmonic in Ω.

An equivalent way to say (b) is:

For all $a \in \Omega$, $\xi \in \mathbb{C}^n$, $|\xi| < d(a, \Omega^c)$, then

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\xi)\, d\theta$$

Thm: For any decreasing sequence of plurisubharmonic function $u_k \in Psh(\Omega)$, then

$$u = \lim_{k \to \infty} u_k \text{ is also in } Psh(\Omega).$$

Proof: Excercise.

Proposition: Let $u \in C^2(\Omega)$. Then $u \in Psh(\Omega)$ iff

$$\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_l}\right)_{1 \leq j, l \leq n} \geq 0 \text{ at every } p \in \Omega.$$

Proof: $u \in Psh(\Omega)$ if and only if for $\forall z_0 \in \Omega$ and $a \in \mathbb{C}^n$,

$\tilde{u}(\lambda) = u(z_0 + \lambda a)$ is subharmonic in λ where it is defined.

This is true if and only if

$$\Delta \tilde{u}(\lambda) = \frac{\partial^2 u(z_0 + \lambda a)}{\partial \lambda \partial \bar{\lambda}} \geq 0 \qquad (*)$$

We compute $\dfrac{\partial^2 u(z_0 + \lambda a)}{\partial \lambda \partial \bar{\lambda}} = \displaystyle\sum_{j,l=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_l} a_j \bar{a_l}$.

$(*)$ is equivalent to $\left(\dfrac{\partial^2 u}{\partial z_j \partial \bar{z}_l}\right)_{1 \leq j, l \leq n} \geq 0$.

Remark: Let $u \in C^2(\Omega)$, $p \in \Omega$, $w \in \mathbb{C}^n$. We will write

$$L_p(u, w) := \sum_{j,l=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_l} w_j \bar{w_l}.$$

It is called the Levi form of u.

Thus $u \in Psh(\Omega)$ iff $L_p(u, w) \geq 0$ for all $p \in \Omega$, $w \in \mathbb{C}^n$.

Defn: Let $u \in C^2(\Omega)$, $p \in \Omega$. If $L_p(u,w) > 0$ for all $w \in \mathbb{C}^n$, we say u is strictly plurisubharmonic at p. If this is true for all $p \in \Omega$, then we say u is strictly plurisubharmonic in Ω.

Notation: $u \in SPsh(\Omega)$.

E.g: If $u = |z|^2$, then

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_l} = \delta_{jl}.$$

Thus $u \in SPsh(\Omega)$.

Proposition: Let G, D be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$. $F: D \to G$ holomorphic map. If $u \in C^2(G) \cap Psh(G)$, then

$$u \circ F \in Psh(D).$$

Proof: Let $a \in D$. Then $L_a(u \circ F, w) = L_{F(a)}(u, F'(a)w)$.

Thm: Let $u \in Psh(\Omega)$. Then $\exists\, u_\varepsilon$ that is C^∞ when $\operatorname{dist}(z, G^c) > \varepsilon$, and $u_\varepsilon \in Psh(\Omega)$ such that $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$

Corollary: Let G, D be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, $F: D \to G$ a holomorphic map. If $u \in Psh(\Omega)$, $u \circ F \in Psh(\Omega)$.

Defn: Let Ω be a domain in $\mathbb{C}^n$, φ a real-valued function in Ω. We say φ is an exhaustion function of Ω if for $\forall c$, $\{z \in \Omega : \varphi(z) \le c\} \subset\subset G$.

Defn: Let Ω be a domain in $\mathbb{C}^n$, we say Ω is pseudoconvex if there exists a continuous plurisubharmonic exhaustion function of Ω.

Thm: Let Ω be pseudoconvex. Then there exists a C^∞ strictly plurisubharmonic exhaustion function of Ω.

Proof: See Thm 2.6.11 in Hormander's book.