Holomorphic convex

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Holomorphic convexity

(I) convexity in the usual sense.

Let G be an open subset of $\mathbb{R}^N$.

- We say G is geometrically convex if:
  for any $z_0 \in \mathbb{R}^N - G$, there exists a real line l passing through z_0 such that $G \subset H_l^-$, where
  $$H_l^- = \{z \in \mathbb{R}^N : l(z) < 0\}.$$

- We say G is convex if
  for any $z_1, z_2 \in G$, and any $0 < \lambda < 1$, we have
  $$\lambda z_1 + (1-\lambda) z_2 \in G.$$

The two notions are equivalent.

proposition: G is geometrically convex $\iff$ G is convex.

Pf: we will only prove for the case $N=2$.

"$\Longrightarrow$" proof by contrapositive.

Assume G is not convex, then $\exists\, z_1, z_2 \in G$ and $0<\lambda<1$ such that
$$z_0 := \lambda z_1 + (1-\lambda) z_2 \notin G$$

For any line $l(z) = ax + by + c$, $a, b, c \in \mathbb{R}$, $z = (x, y)$

Write $z_i = (x_i, y_i)$, $0 \le i \le 1$. Then
$$\begin{aligned} l(z_0) &= ax_0 + by_0 + c \\ &= a(\lambda x_1 + (1-\lambda) x_2) + b(\lambda y_1 + (1-\lambda) y_2) + c \\ &= \lambda l(z_1) + (1-\lambda) l(z_2) \end{aligned}$$

Thus
$$\min\{l(z_1), l(z_2)\} \le l(z_0) \le \max\{l(z_1), l(z_2)\}.$$

Then z_1, z_2 cannot be both in H_l^- for $\forall\, l$.

Hence G is NOT geometrically convex.

"$\Longleftarrow$" Assume G is convex.

$\because$ " $\Longleftarrow$ " Assume G is convex.

Fix $z_0 \notin G$. For any line L passing through z_0, we can find a direction $a \in \mathbb{R}^2$. such that

$$L = \{z : z = z_0 + ta, t \in \mathbb{R}\}.$$

Let
$$L^+ = \{z : z_0 + ta,\ t \geq 0\}$$
$$L^- = \{z : z_0 + ta,\ t \leq 0\}$$

Claim: G intersects with at most one of L^+, L^-

Pf: Otherwise. G intersects both L^+, L^-. Since G is convex, then $z_0 \in G$. This is a contradiction.

It follows from the claim that, by rotating L, one can find some $\tilde{L}$ passing through z_0 such that

$$G \subset H_{\tilde{L}}^-$$

We next introduce a third equivalent notion of convexity. We first introduce the following notion of convex hull.

Let $K \subset\subset \mathbb{R}^N$. Define the convex hull of K as

$$\hat{K} = \{z \in \mathbb{R}^N : L(z) \leq \sup_{\xi \in K} L(\xi),\ \forall \text{ real linear function } L\}$$

E.g. Let $K = \{z_1, z_2\}$. Then

$$\hat{K} = \text{the line segment } \overline{z_1 z_2}.$$

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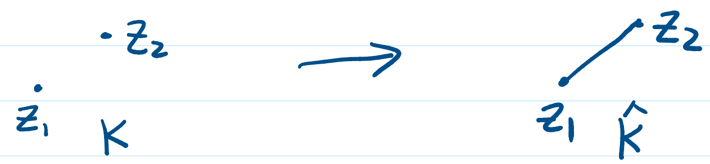

Excercise: For $\forall K \subset\subset \mathbb{R}^N$, prove $\hat{K}$ is compact.

(Hint: prove $\hat{K}$ is bounded + closed)

Proposition: Let G be an open subset.

TFAE (The following are equivalent).

(1) G is geometrically convex.

(2) $\forall K \subset\subset G$, we have $\hat{K} \subset\subset G$.

Proof:

"(1) $\Rightarrow$ (2)" Assume G is geometrically convex.

Let $K \subset\subset G$, we need to show $\hat{K} \subset\subset G$.

Suppose NOT. Then $\exists z_0 \in \hat{K} - G$. Since G is geometrically convex, there exists a real line l passing through z_0 such that $G \subset H_l^-$. That is,

$$l(\xi) < 0 \text{ for any } \xi \in G.$$

In particular, as $K \subset\subset G$, $\Rightarrow$

$$l(\xi) < 0 \text{ for any } \xi \in K.$$

But $l(z_0) = 0$ and $z_0 \in \hat{K}$.

This contradicts the definition of $\hat{K}$.

"(2) $\Rightarrow$ (1)" is easy.

Assume $K \subset\subset G \Rightarrow \hat{K} \subset\subset G$

In particular, let $K = \{z_1, z_2\}$, where $z_1, z_2 \in G$

Recall $\hat{K} = \overline{z_1 z_2}$. As $\hat{K} \subset\subset G$, this implies

$$\lambda z_1 + (1-\lambda) z_2 \in G \text{ for any } 0<\lambda<1.$$

Thus G is convex and thus geometric convex.

Remark: If G is convex and $K \subset\subset G$, then

$$\{z \in \mathbb{R}^N : l(z) \le \sup_{\xi \in K} l(\xi)\} = \{z \in G : l(z) \le \sup_{\xi \in K} l(\xi)\}.$$

We will generalize this idea to the holomorphic setting. We want our notion to carry properties that are invariant under biholomorphic tranformation.

(II) Holomorphic convexity

Let Ω be an open set in $\mathbb{C}^n$ $(n \ge 1)$.

Defn: Let $K \subset \Omega$. We define the holomorphic hull $\hat{K}$ of K as

$$\hat{K}_\Omega = \{z \in \Omega \mid |f(z)| \le \sup_{\xi \in K} |f(\xi)| \text{ for every } f \in H(\Omega)\}.$$

Defn: We say Ω is holomorphically convex if for $\forall K \subset\subset \Omega$, $\hat{K}_\Omega \subset\subset \Omega$

proposition:

(1) $\hat{K}_\Omega \subset \hat{K}$ (convex hull of K)

(2) If K is bounded, then so is $\hat{K}$.

(3) $\hat{K}_\Omega$ is closed in Ω.

(4) $\hat{\hat{K}}_\Omega = \hat{K}_\Omega$

(5) If $K_1 \subset K_2 \subset \Omega$, then $\hat{K}_{1\Omega} \subset \hat{K}_{2\Omega} \subset \Omega$.

proof: (1) In the definition of $\hat{K}_\Omega$, let $L(z)$ be a linear holomorphic function and let

$$f(z) = e^{L(z)}$$

Then $|f(z)| = e^{\operatorname{Re} L(z)}$ and

$$|f(z)| \le \sup_{\xi \in K} |f(\xi)| \iff \operatorname{Re} L(z) \le \sup_{\xi \in K} \operatorname{Re} L(\xi)$$

Then note for a real linear function $l(z)$, there exists a linear holomorphic function $L(z)$ such that

$$l(z) = \operatorname{Re} L(z)$$

(2) In the definition of $\hat{K}_\Omega$. Choose f to be coordinates functions: $f = z_j$, $1 \le j \le n$.

(3), (4), (5) Excercise.