Holomorphy

Domain of holomorphy

Def^n: Ω is called a domain of holomorphy if there does NOT exist open set $U, \hat{\Omega}$, such that

(1) $\phi \neq U \subseteq \Omega \cap \hat{\Omega}$

(2) $\hat{\Omega}$ is connected and $\hat{\Omega} \nsubseteq \Omega$

(3) For every $f \in H(\Omega)$, $\exists\ f_2 \in H(\hat{\Omega})$ s.t. $f = f_2$ in U

Roughly speaking, this means, there is No larger set $\hat{\Omega}$ such that every holomorphic function in Ω can be holomorphically extended to $\hat{\Omega}$.

E.g. Recall: Let $\Omega = B(0,1) \setminus \overline{B(0,\frac{1}{2})}$. Then Ω is NOT a domain of holomorphy, as every $f \in H(\Omega)$ can be holomorphically extended to $B(0,1)$

Defn: (Natural defining domain)

Let $f \in H(\Omega)$. We say Ω is the natural defining domain of f if there does NOT exist open subsets $U, \hat{\Omega}$ and $F \in H(\hat{\Omega})$ s.t.

(1) $\phi \neq U \subset \Omega \cap \hat{\Omega}$

(2) $\hat{\Omega}$ is connected and $\hat{\Omega} \nsubseteq \Omega$

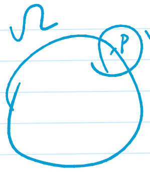
(3) $F = f$ on U.

Roughly speaking, f cannot extends holomorphically to any larger set than Ω.

Remark: Ω is the natural defining domain for some $f \in H(\Omega)$ $\Rightarrow$ Ω is a domain of holomorphy

E.g. Let $\Omega \subset \mathbb{C}$. Then Ω is a domain of holomorphy. If $\Omega = \mathbb{C}$, it is trivial. If not, If $\Omega = \mathbb{C}$, it is trivial. If not,

then for any Ω_2, let $p \in \partial\Omega \cap \Omega_2$.



then $f(z) := \frac{1}{z-p}$ cannot be holomorphically extended to Ω_2.

It is more non-trivial to see Ω is the natural defining domain of some $f \in H(\Omega)$.

Proposition: Let Ω be an open set in $\mathbb{C}^n$. Then

Ω is geometrically convex $\Rightarrow \Omega$ is a domain of holomorphy.

proof: Assume Ω is geometrically convex, if $\Omega = \mathbb{C}^n$, then the conclusion is easy to see. Now assume $\Omega \neq \mathbb{C}^n$. Fix $p \in \partial\Omega$. Since Ω is geometrically convex, there is a real hyperplane l passing through p such that

$$\Omega \subset H_l^-.$$

We can find a holomorphic linear function L such that $l = \mathrm{Re}\, L$. Let

$$L_0 = L - (\mathrm{Im}\, L(p)) \quad \text{Then } \mathrm{Re}\, L_0 = l \text{ and}$$

$$L_0(p) = l(p) = 0$$

Define $f = \dfrac{1}{L_0(z)}$ Note when $z \in \Omega$

we have $\mathrm{Re}\, L_0(z) = l(z) < 0 \Rightarrow f \in H(\Omega)$

But f cannot be extended holomorphically across p.

$\Rightarrow \Omega$ is a domain of holomorphy.

Cartan–Thullen Theorem:

Let $G \subset \mathbb{C}^n$ be an open set. Then TFAE (The following are equivalent):

(i) G is a domain of holomorphy

(ii) G is holomorphically convex

(iii) $\exists f \in H(G)$ s.t. G is the natural defining domain of G

(iii) $\exists f \in H(G)$ s.t G is the natural defining domain of G

(iii)' $\exists f \in H(G)$ s.t f is unbounded near every boundary point of G.

To prove the Cartan-Thullen Thm, we will prove the following Thm.

Notation: Let $z, w \in \mathbb{C}^n$. Define the distance

$$d(z,w) = \max_{1 \le j \le n} |z_j - w_j|$$

Let $A, B \subset \mathbb{C}^n$. Define distance between A, B

$$d(A,B) = \inf\{d(z,w) \mid z \in A, w \in B\}.$$

In particular,

$$d(z,A) = \text{distance between } \{z\}, A$$

Thm 1: Assume G is a domain of holomorphy. Then

$$d(K, G^c) = d(\hat{K}_G, G^c).$$

Here $G^c = \mathbb{C}^n - G$.

Proof: As $K \subset\subset G \Longrightarrow d(K, G^c) > 0$

Fix $0 < r < d(K, G^c)$, let

$K_r = \{z \in \mathbb{C}^n : d(z,K) \le r\}$. Then

$K \subset K_r \subset G$.

Moreover, when $z \in K$, the polydisc

$$P(z,r) = \Delta(z_1,r) \times \cdots \times \Delta(z_n,r) \subset K_r \subset G$$

By Cauchy's Inequality, for $\forall f \in H(G)$, we have

$$|(\partial^\alpha f)(z)| \le \frac{\alpha!}{r^{|\alpha|}} \sup_{\xi \in K_r} |f(\xi)|$$

for all $z \in K$

Fix any $a \in \hat{K}$. By the definition of $\hat{K}$ and note $\partial^\alpha f \in H(G)$ we have

Fix any $a \in \hat{K}$. By the defi... of ... ,

note $\partial^\alpha f \in H(G)$, we have

$$\left|(\partial^\alpha f)(a)\right| \le \frac{\alpha!}{r^{|\alpha|}} \sup_{\xi \in K_r} |f(\xi)| \qquad (*)$$

As K_r is compact, $\sup_{\xi \in K_r} |f(\xi)| = M < \infty$,

Then $(*)$ implies

$$\left| \frac{(\partial^\alpha f)(a)}{\alpha!} (z-a)^\alpha \right| \le M \left(\frac{|z_1 - a_1|}{r}\right)^{\alpha_1} \cdots \left(\frac{|z_n - a_n|}{r}\right)^{\alpha_n}$$

Define

$$g(z) = \sum_{\alpha \ge 0} \frac{(\partial^\alpha f)(a)}{\alpha!} (z-a)^\alpha$$

Note $g(z)$ converges in $P(z,r)$ and defines a holomorphic function in $P(a,r)$ and it agrees with f near a.

Then $\Omega_2 := P(a,r)$ must be contained in G, as G is a domain of holomorphy. This implies

$$d(a, G^c) \ge r. \qquad (**)$$

As $(**)$ holds for all $r < d(K, G^c)$ and $\forall a \in \hat{K}_G$

$$\Rightarrow \qquad d(\hat{K}_G, G^c) \ge d(K, G^c)$$

Note $K \subset \hat{K}_G$ implies $d(\hat{K}_G, G^c) \le d(K, G^c)$

we thus have $d(\hat{K}_G, G^c) = d(K, G^c)$.

Corollary: If G is a domain of holomorphy, then

G is holomorphically convex.

Proof of Cartan-Thullen Thm:

Proof of Cartan-Thullen Thm:

The above Corollary shows (i) $\Rightarrow$ (ii)

Apparently, (iii)' $\Rightarrow$ (iii) $\Rightarrow$ (i). To show the equivalence, we just need to prove (ii) $\Rightarrow$ (iii)'.

This follows from the next Thm.

Theorem 2: Assume G is holomorphically convex, then there exists $f \in H(G)$, such that

f is unbounded near every boundary point of G.

Before proving the thm, we introduce the following definition.

Defn: (exhausting compact sets)

Let G be an open set in $\mathbb{C}^n$. Then $\{K_j\}_{j=1}^{\infty}$ with $K_j \subset G$ is called an exhausting sequence (of compact sets) of G if

(i) each K_j is compact

(ii) $\bigcup_{j=1}^{\infty} K_j = G$

(iii) $K_j \subset \mathring{K}_{j+1}$ for each $j = 1, 2, \ldots$.

Remark: If $\{K_j\}_{j=1}^{\infty}$ is an exhausting sequence (of compact sets) of G, then for $\forall\, K \subset\subset G$, there exists $j_0 \geq 1$ such that $K \subset K_{j_0}$.

Proof: Excercise!

Lemma 1. Let G be an open subset in $\mathbb{C}^n$. Then

(1) There exists an exhausting sequence $\{K_j\}_{j=1}^{\infty}$ of compact sets of G.

(1) There exists an exhausting sequence $\{K_j\}_{j=1}$ of compact sets of G.

(2) If G is additionally holomorphically convex, then we can make $K_j = \hat{K}_j$ for each j.

Proof: We will leave (1) as an excercise!

proof of (2):

By (1), there is an exhausting sequence $\{K_j\}_{j=1}^{\infty}$. Then we modify them to get a new exhaustion sequence $\{K_j^*\}$ satisfying $(\hat{K}_j^*)_G = K_j$. We will construct the K_j^* inductively.

Step1: Let $K_1^* = (\hat{K}_1)_G$ (Recall $\hat{\hat{K}}_G = \hat{K}_G$)

Step2. Assume $K_1^*, \dots, K_l^*$ have been constructed with $(\hat{K}_j^*)_G = K_j^*$, $K_j^* \subset \mathring{K}_{j+1}^*$

We will next construct K_{l+1}^*. Since K_l^* is compact, by the remark above, $K_l^* \subset \mathring{K}_{\lambda(l)}$ for some $\lambda(l) \in \mathbb{Z}^+$.

We set $K_{l+1}^* = (\hat{K}_{\lambda(l)})_G$. Then

$$K_l^* \subset (\mathring{K}_{l+1}^*). \quad \text{clearly } \bigcup_{l=1}^{\infty} K_l^* = G.$$

This proves (2).

Proposition: Let G be an open set in $\mathbb{C}^n$. Let $\{K_j\}_{j=1}^{\infty}$ be an exhausting sequence of G with $(\hat{K}_j)_G = K_j$. Let $\{P_j\}_{j=1}^{\infty}$ be a sequence of points in G such that $P_j \in K_{j+1} - K_j$, $\forall j \geq 1$. Then $\exists f \in H(G)$, s.t $\lim_{j\to\infty} |f(P_j)| = \infty$.

Lemma 2: Let ...

Lemma 2: Let $G \subset \mathbb{C}^n$ be an open set, and $\{K_j\}_{j=1}^{\infty}$ be an exhausing sequence of G. Then $\exists$ $\{p_j\}_{j=1}^{\infty} \subset G$ and an increasing sequence $\{\nu_j\}_{j=1}^{\infty} \subset \mathbb{Z}^+$ such that

(i) $p_j \in K_{\nu_{j+1}} - K_{\nu_j}$

(ii) For every $\xi \in \partial G$ and any neighborhood $U(\xi)$ of ξ, $\{p_j\}_{j=1}^{\infty}$ has infinitely many points in $U(\xi) \cap G$.

proof of Thm 2: It is easy to see Thm 2 follows from the proposition and Lemma 2. We just need to establish proposition and Lemma 2.

proof of the proposition:

We will first inductively construct a sequence of functions $\{f_j\}_{j=1}^{\infty}$ in $H(G)$ satisfying.

(i) $\sup_{K_j} |f_j(z)| < 2^{-j}$, $\forall j \geq 1$

(ii) $|f_j(p_j)| > j + 1 + \sum_{i=1}^{j-1} |f_i(p_j)|$

Step 1: (Construct f_1)

Since $p_1 \notin K_1$ and $(\hat{K_1})_G = K_1$, then $\exists$ $g_1 \in H(\Omega)$ such that

$$|g_1(p_1)| > M, \text{ where } M = \sup_{\xi \in K_1} |f(\xi)|.$$

Choose s such that

$$M < s < g_1(p_1).$$

Set

$$\tilde{g}_1 = \frac{g_1}{s}. \text{ Then } \tilde{g}_1(p_1) > 1 \text{ and}$$

$$|\tilde{g}_1(\xi)| < 1 \text{ on } K_1$$

$$|\hat{g}_1(\xi)| < 1 \text{ on } K_1$$

Then $f_1 = \tilde{g}_1^{\,m}$ for a large m will satify

$$\sup_{\xi \in K_1} |f_1(\xi)| < \frac{1}{2} \quad \text{and}$$

$$|f_1(p)| > 2$$

step2: Assume we have chosen $f_1, \cdots, f_l$ satifying (i), (ii)

We next construct f_{l+1}

Again since $(\widehat{K_{l+1}})_G = K_{l+1}$, and $p_{l+1} \notin K_{l+1}$

there exists g_{l+1} such that

$$|g_{l+1}(p_{l+1})| > M_{l+1}, \text{ where } M_{l+1} := \sup_{\xi \in K_{l+1}} |g_{l+1}(\xi$$

Pick S_{l+1} with $M_{l+1} < S_{l+1} < |g_{l+1}(p_{l+1})|$

Set $\tilde{g}_{l+1} = \dfrac{g_{l+1}}{S_{l+1}}$ and $f_{l+1} = \tilde{g}_{l+1}^{\,m}$.

Then a sufficiently large m will make

(i) $\sup_{K_{l+1}} |f_{l+1}(z)| < 2^{-(l+1)}$

(ii) $|f_{l+1}(p_{l+1})| > (l+1) + 1 + \sum_{i=1}^{l} |f_i(p_{l+1})|$

Finally, we set

$$f = \sum_{j=1}^{\infty} f_j(z)$$

Then it converges on every K_j and thus on every compact set $K \subset G$.

This implies $f \in H(G)$

Moreover,

$$|f(p_j)| \geq |f_j(p_j)| - \sum_{i=1}^{j-1} |f_i(p_j)| - \sum_{i=j+1}^{\infty} |f_i(p$$

By (1), (2), and $p_j \in K_i$ for $i \geq j+1$.

By [illegible]

$$(*) \geq j+1 - \sum_{i=j+1}^{\infty} 2^{-i} \geq j$$

Thus $\lim_{j\to\infty} |f(p_j)| = \infty$.

Proof of Lemma 2: (Sketchy)

First find a dense sequence $\{q_j\}$ in ∂G.

(i) pick $p_1 \in K_2 - K_1$, such that $d(p_1, q_1) < 2^{-1}$.

(ii) pick $p_2 \notin K_2$ such that $d(p_2, q_2) < 2^{-2}$.

There exists $\nu_3 \in \mathbb{Z}^+$, such that $z_2 \in K_{\nu_3} - K_{\nu_2}$, where $\nu_2 :=$

(iii) Then pick $p_3 \notin K_{\nu(2)}$, such that $d(p_3, q_3) < 2^{-3}$.

There exists $\nu(3) \in \mathbb{Z}^+$, such that $z_3 \in K_{\nu(3)} - K_{\nu(2)}$.

Inductively, one can find a sequence $\{p_j\}_{j=1}^{\infty} \subset G$, and $\{\nu_j\}_{j=1}^{\infty} \subset \mathbb{Z}^+$ such that

(1) $d(p_j, q_j) < 2^{-j}$

(2) $p_j \in K_{\nu_{j+1}} - K_{\nu_j}$.

It is easy to see the conclusion in the Lemma holds.