Hartog's extension theorem

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Recall the Cauchy Integral formula in $\mathbb{C}$.

Thm. (Cauchy Integral formula) Let Ω be a bounded domain in $\mathbb{C}$ with smooth boundary. If $f \in C^1(\bar\Omega)$, then for every $a \in \Omega$,

$$f(a) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-a}\,dz + \frac{1}{2\pi i}\int_{\Omega} \frac{1}{z-a}\frac{\partial f}{\partial \bar z}\,dz\wedge d\bar z,$$

Pf. Pick $\varepsilon>0$ such that $D_\varepsilon = \{z : |z-a| \le \varepsilon\} \subset \Omega$.

Set $\Omega_\varepsilon = \Omega - D_\varepsilon$, let $\beta = \frac{f(z)}{z-a}dz$.

As $\frac{1}{z-a}$ is holomorphic in Ω_ε. thus, we have

$$d\beta = \partial\beta + \bar\partial\beta = \bar\partial\beta = \frac{1}{z-a}\frac{\partial f}{\partial \bar z}\,d\bar z\wedge dz$$

Apply Stokes' formula on Ω_ε, we have

$$\int_{\partial\Omega}\beta - \int_{\partial D_\varepsilon}\beta = \int_{\Omega_\varepsilon} d\beta = \int_{\Omega_\varepsilon}\frac{1}{z-a}\frac{\partial f}{\partial \bar z}\,d\bar z\wedge dz \quad (*)$$

Let $\varepsilon\to 0$, we get $\int_{\partial D_\varepsilon}\beta \to 2\pi i f(a)$

The RHS of $(*)$ goes to $\int_\Omega \frac{1}{z-a}\frac{\partial f}{\partial \bar z}\,d\bar z\wedge dz$ as $\frac{1}{z-a}$ in locally integrable at 0.

$\bar\partial$-problem on $\mathbb{C}$

Given $\Omega \subset \mathbb{C}^n$, $f \in C^\infty(\Omega)$, solve $\bar\partial u = f$.

Thm 1. Let Ω be a bounded domain in $\mathbb{C}$, $f \in C^1(\Omega)$ and bounded on Ω. set

$$u(z) = \frac{1}{2\pi i}\int_\Omega \frac{f(\xi)}{\xi - z}\,d\xi\wedge d\bar\xi, \quad z\in\Omega. \quad (1)$$

then $u \in C^1(\Omega)$, and $\frac{\partial u}{\partial \bar z} = f$, $\|u\|_\infty \le C\|f\|_\infty$. for some constant $C>0$.

Remark: $\|u\|_\infty = \sup_\Omega |u|$

Pf. We first extend f to the whole plane $\mathbb{C}$ by let $f(z) = 0$ if $z \notin \Omega$.

Then (1) can be written as

$$u(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(\xi)}{\xi - z}\,d\xi\wedge d\bar\xi = \frac{1}{2\pi i}\int_{\mathbb{C}} f(z+\xi)\frac{d\xi\wedge d\bar\xi}{\xi}$$

As $f\in C^1(\Omega) \Rightarrow u \in C^1(\Omega)$.

Now fix $a \in \Omega$. pick a neighborhood V of a s.t $a \in V \subset\subset \Omega$. Then $\exists\, \psi \in C^\infty(\mathbb{C})$ s.t $\psi|_V \equiv 1$, $\operatorname{supp}\psi \subset \Omega$. If we let

$$u_1(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{(\psi f)(\xi)}{\xi - z}\,d\xi\wedge d\bar\xi$$

$$u_2(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{(1-\psi)f}{\xi - z}\,d\xi\wedge d\bar\xi$$

then $u = u_1 + u_2$. both C^1 in V. Now

Since $(1-\psi)f|_V \equiv 0 \Rightarrow$

$$u_2(z) = \frac{1}{2\pi i}\int_{\mathbb{C}-V} \frac{(1-\psi)f}{\xi - z}\,d\xi\wedge d\bar\xi$$

thus when $z\in V$. the integrand is holomorphic in z and is C^1, $\Rightarrow \frac{\partial u_2}{\partial \bar z} = 0$.

Thus on V, we have

$$\frac{\partial u}{\partial \bar z} = \frac{\partial u_1}{\partial \bar z} = \frac{\partial}{\partial \bar z}\left(\frac{1}{2\pi i}\int_{\mathbb{C}} (\psi f)(z+\xi)\frac{d\xi\wedge d\bar\xi}{\xi}\right)$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial(\psi f)}{\partial \bar z}(z+\xi)\frac{1}{\xi}\,d\xi\wedge d\bar\xi$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial(\psi f)}{\partial \bar z}\frac{1}{\xi - z}\,d\xi\wedge d\bar\xi \quad (2)$$

By thm 1, we have.

RHS of $(2) = (\psi f)(z) = f(z)$

In particular, $\frac{\partial u}{\partial \bar z}(a) = f(a)$.

As Ω is bounded, $\exists$ large $R>0$, s.t for $\forall z\in\Omega$ the disc $\Delta(z,R)$ covers Ω. Now as $d\bar\xi\wedge d\xi = 2i\,dx\wedge dy$.

$\Rightarrow$ for $\forall z \in \Omega$,

$$|u(z)| \le \frac{1}{2\pi}\int_\Omega \frac{|f(\xi)|}{|\xi - z|}|d\lambda\wedge d\bar\lambda|$$

$$\le \frac{\|f\|_\infty}{\pi}\int_{\Delta(z,R)}\frac{dxdy}{r} = 2R\|f\|_\infty.$$

$\bar\partial$-problem in $\mathbb{C}^n$ with compact support.

Defn. Let $f = \sum_{j=1}^n f_j(z)d\bar z_j$ be a $(0,1)$ form in $\mathbb{C}^n$. the union of $\operatorname{supp} f_j$ is called the support of f, i.e.,

$$\operatorname{supp} f = \bigcup_{I,J}\operatorname{supp} f_{IJ}$$

we say f has a compact support, if $\operatorname{supp} f$ is compact

Thm 2. Let $n>1$, $f = \sum_{j=1}^n f_j(z)d\bar z_j$ be a $(0,1)$ form in $\mathbb{C}^n$. $f_j \in C^k(\mathbb{C}^n)$, $j=1,\cdots,n$. Assume f has a compact support K. If $\bar\partial f = 0$ in $\mathbb{C}^n$, then $\exists$ a unique $u\in C^k(\mathbb{C}^n)$ s.t

① $\bar\partial u = f$

② $\operatorname{supp} u$ is compact.

Pf. Set

$$u(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} f_1(\xi, z_2,\cdots,z_n)\frac{d\xi\wedge d\bar\xi}{\xi - z_1}$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} f_1(z_1+\xi, z_2,\cdots,z_n)\frac{d\xi\wedge d\bar\xi}{\xi}$$

As $f_1 \in C^k(\mathbb{C}^n) \Rightarrow u \in C^k(\mathbb{C}^n)$.

By thm 1, $\frac{\partial u}{\partial \bar z_1} = f_1$ Now we need to show

$$\frac{\partial u}{\partial \bar z_j} = f_j \text{ for } j = 2,\cdots,n.$$

As $\bar\partial f = \sum_{j=1}^n\sum_{k=1}^n \frac{\partial f_j}{\partial \bar z_k}d\bar z_k\wedge d\bar z_j = \sum_{k<j}\left(\frac{\partial f_j}{\partial \bar z_k} - \frac{\partial f_k}{\partial \bar z_j}\right)d\bar z_k\wedge d\bar z_j$

thus $\bar\partial f = 0 \Leftrightarrow \frac{\partial f_j}{\partial \bar z_k} = \frac{\partial f_k}{\partial \bar z_j}$ for $\forall j,k$.

Then if $j = 2,\cdots,n$, we have

$$\frac{\partial u}{\partial \bar z_j}(z) = \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\partial f_1}{\partial \bar z_j}(\xi, z_2,\cdots,z_n)\frac{d\xi\wedge d\bar\xi}{\xi - z_1}$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\partial f_j}{\partial \bar z_1}(\xi, z_2,\cdots,z_n)\frac{d\xi\wedge d\bar\xi}{\xi - z_1}$$

By Cauchy Integral formula, $\Rightarrow$

the above $= f_j(z_1, z_2,\cdots,z_n)$

This proves $\bar\partial u = f$.

Let Ω_0 be the unbounded component of $\mathbb{C}^n - K$. In particular, we have $f_j \equiv 0$ on Ω_0, $\forall j$.

Thus $\bar\partial u = 0$, this implies $u \in H(\Omega_0)$.

hen $|z_2|$ is sufficiently large, then

$z = (z_1, z_2, \cdots, z_n) \notin K$. and $f_1(\xi, z_2,\cdots,z_n) = 0$

It follows from the definition of u that $u=0$ when $|z_2|$ is large.

By the connectedness of $\Omega_0 \Rightarrow u \equiv 0$ on Ω_0.

This implies $\operatorname{supp} u$ is compact.

Now we prove the uniqueness. Suppose $u, \hat u$ both satisfy ①, ②.

Then $\bar\partial(u - \hat u) = 0$ Thus $u - \hat u \in H(\mathbb{C}^n)$.

By ②, $u(z) = \hat u(z) = 0$ when $|z|$ is large. Hence $u \equiv \hat u$ on $\mathbb{C}^n$.

Thm 3. (Hartog's extension thm)

Let $n>1$, $\Omega \subset \mathbb{C}^n$ be a domain. Let K be a compact subset of Ω. Assume $\Omega - K$ is connected, then every $f \in H(\Omega - K)$ can be extended holomorphically to Ω.

Pf. Let $f\in H(\Omega - K)$. pick a neighborhood V of K, s.t $K \subset V \subset\subset \Omega$.

Then $\exists\, \varphi \in C^\infty(\mathbb{C}^n)$, s.t $\varphi|_V \equiv 1$ and $K_0 := \operatorname{supp}\varphi \subset\subset \Omega$.

we define $u_0 = (1-\varphi)f$. First u_0 is well-defined on $\Omega - K$.

Secondly u_0 is well-defined in V as $1-\varphi \equiv 0$ and thus $u_0 \equiv 0$ in V.

Thus $u_0 \in C^\infty(\Omega)$. The idea is to find u such that

$h := u_0 + u$ gives the required extension. For that,

we will need $\bar\partial u = g$, where $g = -\bar\partial u_0$ is a $(0,1)$ form.

First $g \in C^\infty_{(0,1)}(\Omega)$.

Secondly $1-\varphi \equiv 1$ on $\Omega - K_0$, thus $g \equiv 0$ on $\Omega - K_0$.

Thus we can extends g to $\mathbb{C}^n$ by defining $g(z) \equiv 0$ if $z \notin \Omega$.

In this way, we make $g \in C^\infty_{(0,1)}(\mathbb{C}^n)$ with compact support in K_0.

Then by thm 2, $\exists\, u \in C^\infty(\mathbb{C}^n)$ satisfying

① $\bar\partial u = g$

② $\operatorname{supp} u$ is compact. In particular, $u \equiv 0$ on the unbounded component Ω_0 of $\mathbb{C}^n - K_0$.

Now consider $h = u_0 + u$. Then

① $h \in C^\infty(\Omega)$

② $\bar\partial h = 0$ on Ω, thus $h \in H(\Omega)$

③ Note $\Omega_0 \cap (\Omega - K_0) \ne \emptyset$. $\Rightarrow \exists\, G \subset \Omega - K_0 \subset \Omega - K$ s.t $u = 0$ on G, thus $h = u_0$ on G.

But on $\Omega - K_0$, $u_0 = (1-\varphi)f = f \Rightarrow h = f$ on G.

Since $\Omega - K$ is connected, and $h \in H(\Omega - K) \Rightarrow h \equiv f$ on $\Omega - K$

This proves the thm.