Defn. Let $G \subset \Omega \subset \mathbb{C}^n$. we say G is relatively compact in Ω if

(i) $\overline{G} \subset \Omega$

(ii) $\overline{G}$ is compact.

In this case, we write $G \subset\subset \Omega$

Thm: Let Ω be a domain in $\mathbb{C}^n$, $f \in H(\Omega)$. Assume $K \subset G \subset\subset \Omega$, where K is compact and G is open. Then it holds that

$$\sup_{z \in K} |(D^\alpha f)(z)| \le C_\alpha \sup_{z \in G} |f(z)|$$

Here C_α is a constant only depending on K, G, α.

Pf. Let $r := d(K, \partial G) = \inf_{\substack{z \in K \\ w \in \partial G}} d(z,w)$

where $d(z,w) = \max_k |z_k - w_k|$.

Fix $a = (a_1, \cdots, a_n) \in K$. Then the polydisc

$$P = \{z \in \mathbb{C}^n : |z_k - a_k| < r,\ k = 1, 2, \cdots, n\}$$

is contained in G. By Cauchy's Inequality,

$$|(D^\alpha f)(a)| \le \sup_{\xi \in P} |f(\xi)| \frac{\alpha!}{r^{|\alpha|}} \le C_\alpha \sup_{z \in G} |f(z)|.$$

Thm (Weierstrass) Let Ω be a domain in $\mathbb{C}^n$, $\{f_k\}$ be a sequence of holomorphic functions on Ω. Assume f converges normally to a function f on Ω. Then $f \in H(\Omega)$. Moreover, for each α, $\{D^\alpha f_k\}$ converges normally to $D^\alpha f$ on Ω.

Remark: "converges normally" mean "converges uniformly on every compact subset".

Pf: Fix $a \in \Omega$. Pick $\vec{\rho} = (\rho_1, \cdots, \rho_n)$ such that $\overline{P}(a, \vec{\rho}) \subset \Omega$.

Apply Cauchy Integral formula to f_k on $P(a, \vec{\rho})$:

$$f_k(z) = \frac{1}{(2\pi i)^n} \int \frac{f_k(\xi)}{\prod (\xi_k - z_k)} d\xi_1 \cdots d\xi_n, \quad z \in P(a, \vec{\rho}) \qquad (1)$$

$$f_k(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 P} \frac{f_k(\xi)}{\prod_{k=1}^{n} (\xi_k - z_k)} d\xi_1 \cdots d\xi_n, \quad z \in P(a, \vec{\rho}) \qquad (1)$$

Since $\partial_0 P$ is compact, $\lim_{k\to\infty} f_k(\xi) = f(\xi)$ uniformly on $\partial_0 P$.

Let $k \to \infty$ in (1), we get

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 P} \frac{f(\xi)}{\prod_{k=1}^{n} (\xi_k - z_k)} d\xi_1 \cdots d\xi_n, \quad z \in P(a, \rho).$$

Thus $f \in H(\Omega)$.

Fix a compact subset $K \subset \Omega$. Pick G s.t $K \subset G \subset\subset \Omega$.

As $\overline{G}$ is compact, $f_k \to f$ uniformly on $\overline{G}$. $\Rightarrow$

for $\forall \varepsilon > 0$, $\exists\, k_0$ s.t for $\forall k > k_0$

$$\sup_{z \in G} |f_k(z) - f(z)| < \varepsilon.$$

thus by the above thm,

$$\sup_{z \in K} |D^\alpha (f_k - f)(z)| \le C_\alpha \sup_{z \in G} |f_k(z) - f(z)| < C_\alpha \varepsilon.$$

This means $D^\alpha f_k$ converges uniformly on K to $D^\alpha f$.

Defn: Let $F = \{f_k\}_{k=1}^{\infty}$ be a sequence of holomorphic functions on Ω. We say F is locally uniformly bounded if for every compact subset $K \subset \Omega$, $\exists\, C$ independent of k, such that

$$|f_k(z)| \le C \quad \text{for } k \ge 1,\ z \in K.$$

Thm: (Montel)

Let F be a sequence of holomorphic functions on Ω. Assume F is locally uniformly bounded. Then F has a normally convergent subsequence.

Assume $\mathcal{F}$ is locally uniformly bounded. Then, $\mathcal{F}$
has a normally convergent subsequence.

Before we prove it, let's recall the following thm:

Arzela-Ascoli Thm: Let $K \subset \mathbb{R}^N$ be compact and $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$ a sequence of continuous functions defined on K. If $\mathcal{F}$ is uniformly bounded and equicontinuous on K. Then there is a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ that converges uniformly to a continuous function f on K.

Remark: $\mathcal{F}$ is said to be equipcontinuous on K if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

if $|x-y| < \delta$, then $|f_k(x) - f_k(y)| < \varepsilon$ for $\forall x, y \in D$ and $\forall f_k \in \mathcal{F}$.

<u>proof of Montel's Thm</u>:

Fix a compact subset K and an open subset G with

$$K \subset G \subset\subset \Omega.$$

Then by the previous thm, we have for every $1 \le j \le n$,

$$\sup_K \left| \frac{\partial f_k}{\partial z_j} \right| \le C_j \sup_G |f_k| \le M_j$$

i.e $\left\{ \frac{\partial f_k}{\partial z_j} \right\}_{k=1}^{\infty}$ is uniformly bounded on K. Consequently, $\left\{ \frac{\partial f_k}{\partial x_j} \right\}_{k=1}^{\infty}$, $\left\{ \frac{\partial f_k}{\partial y_j} \right\}_{k=1}^{\infty}$ are also uniformly bounded on K.

Write $f_k = u_k + i v_k$. then $\left\{ \frac{\partial u_k}{\partial x_j} \right\}$, $\left\{ \frac{\partial u_k}{\partial y_j} \right\}$, $\left\{ \frac{\partial v_k}{\partial x_j} \right\}$, $\left\{ \frac{\partial v_k}{\partial y_k} \right\}$ are uniformly bounded on K. Then by M.V.T

$\{f_k\}$ is equicontinuous on K.

$\{f_k\}$ is equicontinuous on K.

Since $\{f_k\}$ is also uniformly bounded on K

By Arzela-Ascoli thm, $\{f_k\}$ has a uniformly convergent subsequence on K.

Next we need to find a subsequence of $\mathcal{F}$ that uniformly converges on every compact subset of Ω. To do that, we ne[ed] the following fact:

proposition: Let Ω be a domain in $\mathbb{C}^n$. Then $\exists$ a sequ[ence] of compact subsets $\{K_j\}_{j=1}^{\infty}$ of Ω. such th[at]

(1) $K_j \subset \overset{\circ}{K}_{j+1}$, $\forall j \geq 1$

(2) $\bigcup_{j=1}^{\infty} K_j = \Omega$.

Moreover, it follows that for every compact subset K there is $j_0 \geq 1$ such that $K \subset K_{j_0}$.

Pf: Excercise.

Then by the discussion above, for every K_j, we can f[ind] a subsequence of $\{f_k\}$ that uniformly converges on K_j.

Call it $\{f_k^j\}_{k=1}^{\infty}$.

Using Cantor's diagonalization method, we can
a subsequence, $\{f_j^j\}_{j=1}^{\infty}$ that uniformly converges o
K_l. and thus it converges normally on Ω.