We will discuss more properties and applications of the Bergman kernel.

Thm (Fefferman) Let Ω be a bdd strongly pseudoconvex domain with smooth bdry. Let ρ be a smooth defining function of Ω:

$$\begin{cases} \Omega = \{ z \in \mathbb{C}^n : \rho(z) > 0 \}. \\ d\rho(z) \neq 0 \text{ for all } z \in \partial\Omega. \end{cases}$$

Write K for the Bergman kernel of Ω.

Then

$$K(z,\bar{z}) = \frac{\phi(z)}{\rho^{n+1}(z)} + \psi(z) \log \rho(z).$$

Where $\phi, \psi \in C^\infty(\bar{\Omega})$ and $\phi|_{b\Omega} \neq 0$.

Using this result, Klembeck showed that.

Recall: given a Kähler metric $g_{i\bar{j}}$ on a domain Ω.

Let R be the curvature of the Levi-Civita connection.

Let J be an endormorphism $T\Omega \to T\Omega$ by

$$\begin{cases} J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \\ J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i} \end{cases} \quad \forall\, 1 \le i \le n$$

Then extend by linearity.

Let $z \in \Omega$ and $P \subseteq T_z\Omega$ be a 2-plane invariant by J. Let X be a unit vector in P. Then

$$K(z,P) = R(X, JX, X, JX)$$

is called the holomorphic sectional curvature by P.

Thm (Klembeck)

Let $\Omega \subseteq \mathbb{C}^n$, $n \ge 2$, be a bdd strongly pseudoconvex domain with smooth bdry. Let $g_{i\bar{j}}$ be the Bergman metric of Ω. $\Rightarrow$ $K(z,P) \to \frac{-2}{n+1}$ as $z \to \partial\Omega$.

Idea of pf:

Recall $g_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K$

with
$$K = \frac{\phi}{\rho^{n+1}} + \psi \log \rho$$
$$= \rho^{-(n+1)} \left(\phi + \rho^{n+1} \psi \log \rho \right)$$

write $r = \rho \left(\phi + \rho^{n+1} \psi \log \rho \right)^{-\frac{1}{n+1}}$

then $K = r^{-(n+1)} \Rightarrow g_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log r^{-(n+1)}$

This is very similar to the Bergman metric of the unit ball

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \left(1 - |z|^2\right)^{-(n+1)}$$

Then use the formulas

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - g^{s\bar{t}} \frac{\partial g_{s\bar{j}}}{\partial z_k} \frac{\partial g_{i\bar{t}}}{\partial \bar{z}_l} ;$$

and $g_{i\bar{j}}$ has CHSC $= c \left(= \frac{-2}{n+1}\right) \Longleftrightarrow$

$$R_{i\bar{j}k\bar{l}} = \underbrace{\frac{-c}{2}}_{\frac{1}{n+1}} (g_{i\bar{j}} g_{k\bar{l}} + g_{k\bar{j}} g_{i\bar{l}})$$

By direct (by rather complicated) computation, Klembeck show,

$$R_{i\bar{j}k\bar{l}} = \underbrace{\frac{1}{n+1}(g_{i\bar{j}} g_{k\bar{l}} + g_{k\bar{j}} g_{i\bar{l}})}_{\frac{1}{p^2}} + \underbrace{\text{Error}}_{\text{lower order}}$$

Let G be a bdd domain in $\mathbb{C}^n$. Let

$\mathrm{Aut}(G) = \{f : f$ is a self-biholomorphisms of $G\}$.

We say $\mathrm{Aut}(G)$ is noncompact if

(a) $\exists \{\varphi_j\}_{j=1}^{\infty} \subseteq \mathrm{Aut}(G)$ s.t $\{\varphi_j\}$ has no subsequence that converses to an automorphism $\varphi \in \mathrm{Aut}(G)$ normally.

Fact: : (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c)

(b). $\exists\, p \in \Omega$ and $q \in \partial\Omega$, and a sequence $\{\varphi_j\}_{j=1}^{\infty} \subseteq \mathrm{Aut}(G)$ s.t $\varphi_j(p) \to q$ as $j \to \infty$.

(c). $\exists$ a sequence $\{\varphi_j\}_{j=1}^{\infty} \subseteq \mathrm{Aut}(G)$, s.t for $\forall p \in \Omega$, $\varphi_j(p) \to \partial\Omega$ as $j \to \infty$

Remark: It's easy to see (c) $\Rightarrow$ (b) $\Rightarrow$ (a)

To see why (a) $\Rightarrow$ (c), check P147-150 in Krantz's book.

A well-know thm regarding strongly pseudoconvex domain with noncompact automorphism group.

Thm (Bun Wong) Let $G \subseteq \mathbb{C}^n$, $n \geq 2$, be a bdd strongly pseudoconvex domain with smooth bdry. If $\mathrm{Aut}(G)$ is noncompact, then G is biholomorphic to $\mathbb{B}^n$.

Pf: Let $g_{i\bar{j}}$ be the Bergman metric of G.

Write $K(z,p)$ be the holo. sectional curvature of $g_{i\bar{j}}$ with $z \in \Omega$ and $p \in T_z\Omega$ invariant under J.

Fix any $z_0 \in G$, $p_0 \subseteq T_{z_0}G$. Since $Aut(G)$ is noncompact, $\exists$ a sequence $\{\varphi_j\}_{j=1}^{\infty} \subseteq Aut(G)$ s.t $\varphi_j(z_0) \to \partial G$.

By Klembeck, $K(\varphi_j(z_0), p_j) \to \frac{-2}{n+1}$.

with $p_j \in T_{\varphi_j(z_0)}\Omega$.

Take $p_j = d\varphi_j(p_0)$. Since $\varphi_j \in Aut(G) \Rightarrow$

$$K(z_0, p_0) = K(\varphi_j(z_0), p_j).$$

$\Rightarrow K(z_0, p_0) = \frac{-2}{n+1}$. $\forall z_0 \in G$, $p_0 \subseteq T_{z_0}G$ invariant under J

Hence $(G, g_{i\bar{j}})$ has CHSC $= \frac{-2}{n+1}$.

Then By uniformization thm of Lu $\Rightarrow$

G is biholo. to $\mathbb{B}^n$.

Corollary: The only bdd strongly pseudoconvex domain with smooth bdry in $\mathbb{C}^n$, $n \geq 2$, that can cover a compact complex manifold is $\mathbb{B}^n$.