Recall the uniformization thm in one complex variable:

<u>Def^n</u>: A Riemann surface is a connected Haudorff space M together with a collection of charts $\{U_\alpha, z_\alpha\}$ satisfying

1. The U_α form an open covering of M
2. Each z_α is a homeomorphic mapping of U_α onto an open subset of $\mathbb{C}$
3. If $U_\alpha \cap U_\beta \neq \phi$, then $f_{\alpha\beta} = z_\beta \circ z_\alpha^{-1}$ is holomorphic on $z_\alpha(U_\alpha \cap U_\beta)$

(Uniformization Thm for Riemann surface, Koebe, Poincaré, 1907):

Every simple connected Riemann surface is biholomorphic to one of the following:

1. the Riemann sphere
2. $\mathbb{C}$
3. $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

However the analog in higher dimension fails dramatically.

However, the analog

E.g 1: $\mathbb{B}^n$ and Δ^n are NOT biholomorphic (Poincaré)

E.g 2 (Burns, Shnider, Wells, 1978)

Among small perturbations of $\mathbb{B}^n$, $n \geq 2$, there are infinitely many mutually non-biholomorphic domains. Consequently, there exist infinitely many mutually non-biholomorphic complex structures on $\mathbb{B}^n$

Therefore to formulate a uniformization thm in higher dimension, one has to put stronger assumptions.

A classical uniformization thm:

Let M be a simply connected complete Kähler manifold M of constant holomorphic sectional curvature c.
Then M is holomorphic isometric to one of the following:

1. $(\mathbb{CP}^n, \lambda \omega_{FS})$ for some $\lambda > 0$
2. $(\mathbb{C}^n, \omega_E)$
3. $(\mathbb{B}^n, \lambda \omega_B)$ for some $\lambda > 0$

we will prove a closely related uniformization result in terms of the Bergman metric

Thm (Qikeng Lu) Let D be an bdd domain in $\mathbb{C}^n$. Then D has complete Bergman metric of constant holomorphic sectional curvature iff D is biholomorphic to the unit ball.

We first explain a bit the "holomorphic sectional curvature".

Let $(\Omega, g_{i\bar{j}})$ be a domain equipped with a Kähler metric. Let R be the curvature of the Levi-Civita connection: for vector fields X, Y, Z, W

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$R(X,Y,Z,W) = -g(R(X,Y)Z, W).$$

Let J be an endomorphism $TM \to TM$ by

$$\begin{cases} J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \\ J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i} \end{cases} \quad \forall i$$

Then extends by linearity.

Thm. Let P be a 2-plane invariant by J. Let X be a unit vector in P. Then

$$K(P) \triangleq R(X, JX, X, JX)$$

is called the holomorphic sectional curvature by P

Remark: · $K(P)$ does not depend on the choice of P.

· Let $U = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX)$. Then

$$K(P) = -R(U, \bar{U}, U, \bar{U}).$$

Def^n: If $K(P)$ is constant for all plane P in T_xM invariant by J and for all points $x \in M$, then M is called a space of constant holomorphic sectional curvature.

In the coordinates $z=(z_1, \cdots, z_n)$, we write

$$R_{i\bar{j}k\bar{l}} = R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right).$$

Remark: • We have the following formula for

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - g^{s\bar{t}} \frac{\partial g_{s\bar{j}}}{\partial z_k} \frac{\partial h_{i\bar{t}}}{\partial \bar{z}_l}$$

• $(\Omega, g_{i\bar{j}})$ has constant holomorphic sectional curvature $= c$ $\iff$

$$R_{i\bar{j}k\bar{l}} = -\frac{c}{2}\left(g_{i\bar{j}}g_{k\bar{l}} + g_{k\bar{j}}g_{i\bar{l}}\right) \qquad (*)$$

E.g. Let $(\mathbb{B}^n, g_{i\bar{j}})$ be the unit ball equipped with the Bergman metric

$$g_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j}(\log K)$$

Then $(\mathbb{B}^n, g_{i\bar{j}})$ has constant holo. sectional curvature $= -\frac{2}{n+1}$.

Pf: Since $(\mathbb{B}^n, g_{i\bar{j}})$ homogeneous, it suffices to check at 0.

By a standard computation,

$$g_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log (1-|z|^2)^{-(n+1)}$$

$$= (n+1) \frac{(1-|z|^2)\delta_{ij} + \bar{z}_i z_j}{(1-|z|^2)^2}$$

$\Rightarrow$ At $z=0$, $\quad g_{i\bar{j}}(0) = (n+1)\delta_{ij}$

Note:

$$g_{i\bar{j}} = (n+1)\left(\delta_{i\bar{j}}(1+|z|^2) + \bar{z}_i z_j\right) + O(4)$$

$\Rightarrow$

$$\begin{cases} \dfrac{\partial g_{i\bar{j}}}{\partial z_k}(0) = 0 \\[2ex] \dfrac{\partial g_{i\bar{j}}}{\partial \bar{z}_l}(0) = 0 \end{cases}$$

$$\frac{\partial g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}(0) = (n+1)\left(\delta_{ij}\delta_{kl} + \delta_{jk}\delta_{il}\right)$$

Recall

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - g^{s\bar{t}} \frac{\partial g_{s\bar{j}}}{\partial z_k} \frac{\partial h_{i\bar{t}}}{\partial \bar{z}_l}$$

$\Rightarrow$

$$R_{i\bar{j}k\bar{l}}\Big|_0 = (n+1)\left(\delta_{i\bar{j}}\delta_{k\bar{l}} + \delta_{\bar{j}k}\delta_{i\bar{l}}\right)$$

$$= \frac{1}{n+1}\left(g_{i\bar{j}} g_{k\bar{l}} + g_{k\bar{j}} g_{i\bar{l}}\right)\Big|_0$$

$$= -\frac{c}{2}\left(g_{i\bar{j}} g_{k\bar{l}} + g_{k\bar{j}} g_{i\bar{l}}\right)\Big|_0$$

if we let $c = -\frac{2}{n+1}$. $\boxed{///}$

E.x: Let $h_{i\bar{j}} = \frac{\lambda}{n+1} g_{i\bar{j}}$, $\lambda > 0$, be the scaled Bergman metric on $\mathbb{B}^n$. prove $(\mathbb{B}^n, h_{i\bar{j}})$ has constant holo. sectional curvature $= \frac{-2}{\lambda}$.

E.x: let $\mathbb{B}_r^n = \{z \in \mathbb{C}^n : |z| < r\}$

① prove $K_{\mathbb{B}_r^n}(z, \overline{\xi}) = C(n,r)\left(1 - r^{-2}\langle z, \xi\rangle\right)^{-(n+1)}$

② let $h_{i\bar{j}} = \lambda \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\left(1 - r^{-2}\langle z, \xi\rangle\right)^{-1}$

Show $(\mathbb{B}_r^n, h_{i\bar{j}})$ has constant holo. sectional

Curvature $= \frac{-2}{\lambda}$.

We now prove Qikeng Lu's thm, which we repeat below:

Thm (Qikeng Lu) Let D be an bdd domain in $\mathbb{C}^n$. Then D has complete Bergman metric of constant holomorphic sectional curvature iff D is biholomorphic to the unit ball.

Pf: We only need to prove "$\Rightarrow$".

Write the Bergman metric of D as $g_{i\bar{j}}$.

Assume it has constant holo. sectional curvature $= c$. We first make the following observation.

Claim: We must have $c < 0$.

Pf: otherwise, $c = 0$, or $c > 0$.

By the uniformazition thm,

① Suppose $c > 0$. Then D is universally (holomorphically) covered by $\mathbb{CP}^n$. This is impossible as $\mathbb{CP}^n$ is compact and every holo. function on $\mathbb{CP}^n$ is constant.

② Suppose $c = 0$. Then D is universally (holomorphically) covered by $\mathbb{C}^n$. This is impossible by Liouville's Thm.

Hence we must have $c < 0$.

Therefore we have D is universally covered by $\mathbb{B}^n$. (while our goal is to prove $D \simeq \mathbb{B}^n$)

Step 1: Let U be an open subset of $\mathbb{C}^n$ and $p \in U$. Let $z = (z_1, \cdots, z_n)$ be the coordinates, and let $(U, h_{i\bar{j}})$ be a real analytic Kähler metric. Assume

$$h_{i\bar{j}}(z) = \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z, \bar{z})$$

Where $\phi(z, \bar{\xi})$ is real analytic for $(z, \xi) \in U \times U$.

Define $(h^{k\bar{l}})$ be the inverse of $(h_{i\bar{j}})$ in the sense that

$$h_{i\bar{j}} h^{k\bar{j}} \triangleq \sum_{j=1}^{n} h_{i\bar{j}} h^{k\bar{j}} = \delta_{ik}$$

$$\left(\Rightarrow \qquad h_{i\bar{j}} h^{i\bar{l}} = \delta_{jl} \right)$$

Define $\quad w_i(z) = h^{i\bar{l}}(p)\left[\frac{\partial \phi}{\partial \bar{\xi}^l}(z,\bar{p}) - \frac{\partial \phi}{\partial \bar{\xi}^l}(p,\bar{p})\right]$

which is well-define on U.

Note: ① $w_i(p) = 0$

② $$\frac{\partial w_i}{\partial z_k} = h^{i\bar{l}}(p)\,\frac{\partial^2 \phi}{\partial z_k \partial \bar{\xi}_l}(z,\bar{p})$$

$$\left.\frac{\partial w_i}{\partial z_k}\right|_p = h^{i\bar{l}}(p)\, h_{k\bar{l}}(p) = \delta^i_k$$

Hence at p, $z = (z_1, \cdots, z_n) \to w = (w_1, \cdots, w_n)$ is a (biholomorphic) change of coordinates in a possibly small nbhd V of p.

Such (w, V) is called the representive coordinates chart at p

<u>Lemma 1</u>: Let (w, V) be the representive coordinates chart at p. Set

$$\hat{\Phi}(w, \bar{w}) = \Phi(z(w), \overline{z(w)})$$

the metric now in terms of w-coordinates, is given by

$$\hat{h}_{i\bar{j}}(w, \bar{w}) = \frac{\partial^2 \hat{\Phi}(w, \bar{w})}{\partial w_i \, \partial \bar{w}_j}$$

$$= \frac{\partial^2 \Phi(z, \bar{z})}{\partial z_k \, \partial \bar{z}_l} \frac{\partial z_k}{\partial w_i} \overline{\frac{\partial z_l}{\partial w_j}}$$

$$= h_{k\bar{l}}(z, \bar{z}) \frac{\partial z_k}{\partial w_i} \overline{\frac{\partial z_l}{\partial w_j}}$$

It satisfies that:

$$\hat{h}_{i\bar{j}}(w, 0) \equiv h_{i\bar{j}}(p, \bar{p}), \ \forall w \in V$$

$$\Rightarrow \left. \frac{\partial^2 \hat{\Phi}(w, \eta)}{\partial w_i \, \partial \bar{\eta}_j} \right|_{(w, \eta) = (w, 0)} \equiv \text{constant}, \ \forall w \in V$$

Consequently, for $k_1+\cdots+k_n \geq 2$, $k_i \geq 0$

$$\left[\frac{\partial^{k_1+\cdots+k_n+1}\hat{\Phi}(w,0)}{(\partial w_1)^{k_1}\cdots(\partial w_n)^{k_n}(\partial \overline{\eta}_j)}\right] = 0, \ \forall w \in V$$

In particular

$$\left[\frac{\partial^{k_1+\cdots+k_n+1}\hat{\Phi}(0,0)}{(\partial w_1)^{k_1}\cdots(\partial w_n)^{k_n}(\partial \overline{\eta}_j)}\right] = 0, \ .$$

Remark: For $\alpha=(\alpha_1,\cdots,\alpha_n)$, $\beta=(\beta_1,\cdots,\beta_n)$

write $D^{\alpha}D^{\overline{\beta}}\hat{\Phi} = \dfrac{\partial^{|\alpha|+|\beta|}\hat{\Phi}}{\partial z^{\alpha}\partial \overline{z}^{\beta}}$

$\Rightarrow$ $D^{\alpha}D^{\overline{\beta}}\hat{\Phi}(0) = 0$ if $\begin{cases}|\alpha|\geq 2\\ |\beta|=1\end{cases}$

or $\begin{cases}|\alpha|=1\\ |\beta|\geq 2\end{cases}$

Remark: The Taylor expansion of $\hat{\Phi}$ at 0 takes the form:

$$\hat{\Phi}(w,\bar{w}) = \sum_{\alpha,\beta} D^{\alpha} D^{\bar{\beta}} \hat{\Phi}(0)\, w^{\alpha}\, \overline{w^{\beta}}$$

$$= 2 \cdot \frac{\hat{\Phi}(0,0)}{2} + \sum_{\substack{\alpha=0\\ \beta\geq 1}} D^{\bar{\beta}} \hat{\Phi}(0)\, \overline{w^{\beta}}$$

$$+ \sum_{\substack{\alpha\geq 1\\ \beta=0}} D^{\alpha} \hat{\Phi}(0)\, w^{\alpha} + \sum_{\substack{|\alpha|=1\\ |\beta|=1}} D^{\alpha} D^{\beta} \hat{\Phi}(0)\, w^{\alpha}\, \overline{w^{\beta}}$$

$$\sum_{\substack{|\alpha|=1\\ |\beta|\geq 2}} D^{\alpha} D^{\bar{\beta}} \hat{\Phi}(0)\, w^{\alpha}\, \overline{w^{\beta}} + \sum_{\substack{|\alpha|\geq 2\\ |\beta|=1}} D^{\alpha} D^{\bar{\beta}} \hat{\Phi}(0)\, w^{\alpha}\, \overline{w^{\beta}}$$

$$+ \sum_{\substack{|\alpha|\geq 2\\ |\beta|\geq 2}} D^{\alpha} D^{\bar{\beta}} \hat{\Phi}(0)\, w^{\alpha}\, \overline{w^{\beta}}$$

$$= f(w) + \overline{f(w)} + |w|^2 + \sum_{\substack{\alpha \geq 2 \\ \beta \geq 2}} D^\alpha D^{\overline{\beta}} \hat{\Phi}(0) w^\alpha \overline{w^\beta}$$

Replacing $\hat{\Phi}$ by $\hat{\Phi} - (f(w) + \overline{f(w)})$, we can assume

$$\hat{\Phi} = |w|^2 + \sum_{\substack{\alpha \geq 2 \\ \beta \geq 2}} D^\alpha D^{\overline{\beta}} \hat{\Phi}(0) w^\alpha \overline{w^\beta} \qquad (\triangle)$$

Pf of Lemma 1: Set

$$h_{k\bar{l}}(z, \overline{\xi}) = \frac{\partial^2 \Phi(z, \overline{\xi})}{\partial z_k \, \partial \overline{\xi}_l}, \quad (z, \xi) \in U \times U$$

Then

$$h_{k\bar{l}}(z, \overline{\xi}) \Big|_{(z, \overline{\xi}) = (z, \overline{z})} = \frac{\partial^2 \Phi(z, \overline{z})}{\partial z_k \, \partial \overline{z}_l}.$$

Set $\hat{\Phi}(w, \bar{\eta}) = \Phi(z(w), \overline{\xi(\eta)})$, $w, \eta \in V$

$$\hat{h}_{i\bar{j}}(w, \bar{\eta}) = \frac{\partial^2 \hat{\Phi}(w, \bar{\eta})}{\partial w_i \, \partial \bar{\eta}_j}$$

$$= \frac{\partial^2 \Phi(z, \bar{\xi})}{\partial z_k \, \partial \bar{\xi}_l} \frac{\partial z_k}{\partial w_i}(z) \overline{\frac{\partial \xi_l}{\partial \eta_j}(\xi)}$$

$$= h_{k\bar{l}}(z(w), \overline{\xi(\eta)}) \cdot \frac{\partial z_k}{\partial w_i}(w) \overline{\frac{\partial \xi_l}{\partial \eta_j}(\eta)} \qquad (*)$$

Let $(h^{i\bar{j}}(z, \bar{\xi}))$ be s.t

$$h^{i\bar{l}}(z, \bar{\xi}) \, h_{k\bar{l}}(z, \bar{\xi}) = \delta_k^i,$$

Note, by shrinking V if necessary, we can assume each $h^{i\bar{j}}(z, \bar{\xi})$ is holomorphic in $(z, \bar{\xi}) \in V \times \bar{V}$.

Claim 1: $\frac{\partial z_k}{\partial w_i}(w) = h^{k\bar{l}}(z(w), \bar{p}) \, h_{i\bar{l}}(p, \bar{p})$

Likewise, $\frac{\partial \xi_l}{\partial \eta_j}(\eta) = h^{l\bar{k}}(\xi(\eta), \bar{p}) \, h_{j\bar{k}}(p, \bar{p})$

Pf: Recall

$$\frac{\partial w_i}{\partial z_p}(z) = h^{i\bar{l}}(p,\bar{p})\, h_{p\bar{l}}(z,\bar{p}).$$

Note $\quad \dfrac{\partial z_k}{\partial w_i}(w) \cdot \dfrac{\partial w_i}{\partial z_p}(z(w)) = \delta_p^k$

Then the conclusion follows. $\boxed{}$

Now by $(*)$, $\Rightarrow$

$$\hat{h}_{i\bar{j}}(w,\bar{\eta})$$

$$= h_{k\bar{l}}(z(w),\overline{\xi(\eta)})\, h^{k\bar{m}}(z(w),p)\, h_{i\bar{m}}(p,\bar{p})\, \overline{h^{l\bar{q}}(\xi(\eta),\bar{p})\, h_{j\bar{q}}(p,\bar{p})}$$

let $y=0 \Rightarrow \xi = p$, and thus

$$\begin{aligned}
\hat{h}_{i\bar{j}}(w,0) &= h_{k\bar{l}}(z(w),p)\, h^{k\bar{m}}(z(w),p)\, h_{i\bar{m}}(p,\bar{p})\, \overline{h^{l\bar{q}}(p,\bar{p})\, h_{j\bar{q}}(p,\bar{p})} \\
&= \delta_l^m\, h_{i\bar{m}}(p,\bar{p})\, h^{q\bar{l}}(p,\bar{p})\, h_{q\bar{j}}(p,\bar{p}) \\
&= h_{i\bar{l}}(p,\bar{p})\, h^{q\bar{l}}(p,\bar{p})\, h_{q\bar{j}}(p,\bar{p}) \\
&= \delta_i^q\, h_{q\bar{j}}(p,\bar{p}) \\
&= h_{i\bar{j}}(p,\bar{p}) \qquad \boxed{\text{证}}
\end{aligned}$$

Note: For $\forall\, \lambda > 0$,

$\Psi(w,\bar{w}) \triangleq \lambda \log(1-\lambda^{-1}|w|^2)^{-1}$ also satisfies something like (Δ):

$$\Psi(w,\bar{w}) \triangleq \lambda \log(1-\lambda^{-1}|w|^2)^{-1} = |w|^2 + \sum_{\substack{|\alpha|\geq 2 \\ |\beta|\geq 2}} C_{\alpha\bar{\beta}}\, w^\alpha \overline{w^\beta}$$

We will show $\hat{\Phi}(w,\bar{w}) = \Psi(w,\bar{w})$ near $w=0$.

for some appropriate λ.

Lemma 2: Let $(U, h_{i\bar{j}}(z))$ be a Kähler metric as above. Let (w, V) be the representive coordinates at $p \in U$, i.e., for $1 \le i \le n$,

$$w_i(z) = h^{i\bar{l}}(p)\left[\frac{\partial \phi}{\partial \bar{\xi}^l}(z, \bar{p}) - \frac{\partial \phi}{\partial \bar{\xi}^l}(p, \bar{p})\right]$$

Let $\hat{\Phi}$ be has in Lemma 1,

In particular,

$$(1)\quad \begin{cases} D^\alpha \bar{D}^\beta \hat{\Phi}(0) = 0 \text{ if } \begin{cases} |\alpha| \ge 2 \\ |\beta| = 1 \end{cases} \text{ or } \begin{cases} |\alpha| = 1 \\ |\beta| \ge 2 \end{cases} \\ \hat{\Phi}(w, \bar{w}) = |w|^2 + \sum_{\substack{|\alpha| \ge 2 \\ |\beta| \ge 2}} D^\alpha \bar{D}^\beta \hat{\Phi}(0) w^\alpha \bar{w}^\beta \end{cases}$$

Suppose $(U, h_{i\bar{j}}(z))$ (equivalent, $(V, \hat{h}_{i\bar{j}}(w))$)

has constant holo. sectional curvature $= \frac{-2}{\lambda}$

with $\lambda > 0$. Then

$$\hat{\Phi}(w, \bar{w}) = \Psi(w, \bar{w}) \triangleq \lambda \log(1 - \lambda^{-1}|w|^2)^{-1}$$

near $w = 0$.

Pf: Since $(V, \hat{h}_{i\bar{j}}(w))$ has constant holo. sectional curvature $= \frac{-2}{\lambda}$ $\Rightarrow$ Its curvature satisfies

$$R_{i\bar{j}k\bar{l}} = \frac{1}{\lambda}(\hat{h}_{i\bar{j}}\hat{h}_{k\bar{l}} + \hat{h}_{k\bar{j}}\hat{h}_{i\bar{l}})$$

where $R_{i\bar{j}k\bar{l}} = \frac{\partial^2 \hat{h}_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} - \hat{h}^{s\bar{t}} \frac{\partial \hat{h}_{s\bar{j}}}{\partial w_k} \frac{\partial \hat{h}_{i\bar{t}}}{\partial \bar{w}_l}$

This implies, since $\hat{h}_{i\bar{j}} = \frac{\partial^2 \hat{\Phi}(w, \bar{w})}{\partial w_i \partial \bar{w}_j}$

$$\frac{\partial^4 \hat{\Phi}(w,\bar{w})}{\partial w_i \partial w_k \partial \bar{w}_j \partial \bar{w}_l} = \underbrace{\hat{h}^{s\bar{t}} \frac{\partial \hat{h}_{s\bar{j}}}{\partial w_k} \frac{\partial \hat{h}_{i\bar{t}}}{\partial \bar{w}_l} + \frac{1}{\lambda}\left(\hat{h}_{i\bar{j}} \hat{h}_{k\bar{l}} + \hat{h}_{k\bar{j}} \hat{h}_{i\bar{l}}\right)}_{P\left(D^\alpha \hat{\Phi}(w,\bar{w}),\ |\alpha| \le 3\right)} \qquad (1)$$

On the other hand, letting $\mu = \sqrt{\lambda}$, $\mathbb{B}^n_\mu = \{w \in \mathbb{C}^n : |w| < \mu\}$.

$$h_{i\bar{j}}(w,\bar{w}) \triangleq \lambda \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\left(1 - \lambda^{-1}|w|^2\right)^{-1}$$

Then by previous Calculation/E.X, we have $(\mathbb{B}^n_\mu, h_{i\bar{j}})$ has constant holo. sectional curvature $\frac{-2}{\lambda}$. Therefore with

$$\Psi(w,\bar{w}) = \lambda \log\left(1 - \lambda^{-1}|w|^2\right)^{-1}.$$

we have (1) holds for Ψ:

$$\frac{\partial^4 \Psi(w,\bar{w})}{\partial w_i \partial w_k \partial \bar{w}_i \partial \bar{w}_l} = \underbrace{h^{s\bar{t}} \frac{\partial h_{s\bar{j}}}{\partial w_k} \frac{\partial h_{i\bar{t}}}{\partial \bar{w}_i} + \frac{1}{\lambda}\left(h_{i\bar{j}} h_{k\bar{l}} + h_{k\bar{j}} h_{i\bar{l}}\right)}_{P\left(D^\alpha \Psi(w,\bar{w}),\ |\alpha| \le 3\right)} \qquad (3)$$

We now compare $\hat{\Phi}$ and Ψ:

Note: $$\hat{\Phi}(w,\bar{w}) = |w|^2 + O(|w|^4)$$

$$\Psi(w,\bar{w}) = |w|^2 + O(|w|^4)$$

$\Rightarrow$ $\hat{\Phi}$ and Ψ agree up to order 3.

Moreover,

$$\hat{\Phi}(w,\bar{w}) = |w|^2 + \sum_{\substack{|\alpha|\geq 2\\ |\beta|\geq 2}} D^\alpha D^\beta \hat{\Phi}(0)\, w^\alpha \overline{w^\beta}$$

$$\Psi(w,\bar{w}) = |w|^2 + \sum_{\substack{|\alpha|\geq 2\\ |\beta|\geq 2}} D^\alpha D^\beta \Psi(0)\, w^\alpha \overline{w^\beta}$$

But by (1), (2) $\Rightarrow$

$$D^\alpha D^\beta \hat{\Phi}(0) = D^\alpha D^\beta \Psi(0), \text{ for } |\alpha| \geq 2, \ |\beta| \geq 2.$$

Hence $\hat{\Phi} = \Psi$ near 0.

Summerize: At 0, with the charge of coordinates:

$$w_i(z) = h^{i\bar{l}}(p)\left[\frac{\partial \phi}{\partial \bar{\xi}_l}(z, \bar{p}) - \frac{\partial \phi}{\partial \bar{\xi}_l}(p, \bar{p})\right], \ 1 \leq i \leq n$$

$w(z) = (w_1(z), \cdots, w_n(z))$ is a well-defined holomorphic map on U, where $\phi(z, \bar{\xi})$ is holomorphic in $(z, \bar{\xi}) \in U \times \bar{U}$. We proved near p,

$z \to w(z)$ is indeed an isometry.

Step 2:

We next recall an isometry extension Thm from Riemannian geometry.

Thm: Let (M, g) and (N, h) be analytic real manifolds equipped with analytic and complete Riemannian metrics g and h, respectively. Let φ be an isometry of a domain $U \subseteq M$ onto a domain in N. Then φ extends real analytically along any path in M, (image stays in N) and the extension of φ is a local isometry at every pt

If we apply the above isometry extension thm to our setting, the local isometric map $z \to w(z)$ near p, extends along any path in Ω and the extension is holomorphic. This gives a holomorphic multi-valued map from Ω to $\mathbb{B}^n_\mu$.

Note: If Ω is simply connected, then this is indeed single-valued.

We can also consider the local inverse map of $g = w(z)$ from $\mathbb{B}^n_\mu$ to Ω, call it $f = g^{-1}$. Then applying the above isometry extension to f, since $\mathbb{B}^n$ is simply connected, we get a single-valued map from $\mathbb{B}^n$ to Ω. This is indeed the covering map from $\mathbb{B}^n$ to Ω.

Step 3: Now back to the Bergman metric setting. In our setting, we can let

$$\phi(z, \overline{\xi}) = \log K(z, \overline{\xi})$$

Fix $p \in \Omega$. let

$$w_i(z) = h^{i\bar{l}}(p)\left[\frac{\partial \phi}{\partial \bar{\xi}_l}(z,\bar{p}) - \frac{\partial \phi}{\partial \bar{\xi}_l}(p,\bar{p})\right]$$

$w_i(z)$ is indeed well-define on

$$W = \{z \in \Omega;\ K(z,\bar{p}) \neq 0\}$$

($W \neq \phi$ as $p \in K$

The complement of W

$\Omega - W$ is a complex variety of Ω)

Write this map as (w, W)

By step 2, $\exists$ a small nbhd V of p with

$V \subseteq W$ s.t the restriction (w, V) of w

to V is a biholomorphic isometry from V

to $w(V) \subseteq \mathbb{B}^n_\mu$. Moreover, for any path γ in Ω, (w, V) extends holomorphically along γ (with image in $\mathbb{B}^n_\mu$). In particular, if $\gamma \subseteq W$, then the map obtained by the extension is just the restriction of (w, W). This implies, w maps W to $\mathbb{B}^n_\mu$. In particular, w is bdd on W.

We next recall:

Thm (Riemann removable singularity)

Let $D \subseteq \mathbb{C}^n$ be a domain and $g \in H(D)$, $g \not\equiv 0$, write $Z_g = \{z \in D : g(z) = 0\}$.

Assume $f \in H(D - Z_g)$ and bdd on $D - Z_g$.

Then f extends to a holo. function on D.

Apply this removable singularity thm

to $W(z)$ which holo. and bdd on

$W = \Omega - Z_g$, where

$$g = K(z, \overline{p})$$

$$Z_g = \{ z \in \Omega : K(z, \overline{p}) = 0 \}.$$

Hence $W(z)$ extends to a holo. map

defined on Ω. Call it f.

Moreover, f maps Ω to $\mathbb{B}^n_\mu$. (why?)

Summarize:

- We have a global holo. map f from Ω to $\mathbb{B}^n_\mu$, near p, w is a local biholomorphic map

- Nearby p, f has a local inverse h defined near $f(p)$. Note: h is a local isometry at $f(p)$. By the isometry extension thm and the fact that $\mathbb{B}^n_\mu$ is simply-connected, $\Rightarrow$ g extends a global holo. map from $\mathbb{B}^n_\mu$ to Ω.

By uniquenesss of analytic functions

$$\Rightarrow \quad f \circ g = \mathrm{Id}_{\mathbb{B}^n_\mu} \text{ on } \mathbb{B}^n_\mu$$

$$g \circ f = \mathrm{Id}_\Omega \text{ on } \Omega$$

$\Rightarrow$ Ω and $\mathbb{B}^n_\mu$ are biholomorphic.