In this note, we discuss the Bergman metric.

Def^n: For any bdd domain $\Omega \subseteq \mathbb{C}^n$, we define a Hermitian metric at every pt of D by

$$g_{i\bar{j}}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z, \bar{z}), \quad z \in \Omega.$$

This means, at $z \in \Omega$, the tangent vector $\xi = (\xi_1, \cdots, \xi_n)$ is given by

$$|\xi|_{B,z} \triangleq \left(\sum_{i,j=1}^{n} g_{i\bar{j}}(z) \xi_i \bar{\xi}_j \right)^{\frac{1}{2}}$$

This metric is called the Bergman metric.

Remark: We indeed need to show it is actually a metric. That is, $(g_{i\bar{j}}(z))$ is positive definite. we will prove it later.

Under the Hermitian metric $(g_{i\bar{j}}(z))$, the length of a C^1 curve $\gamma: [0,1] \to \Omega$ is given by

$$L(\gamma) = \int_0^1 \left(\sum_{i,j=1}^{n} g_{i\bar{j}}(z)\, \gamma_i'(t)\, \overline{\gamma_j'(t)} \right)^{1/2} dt$$

If $P, Q \in \Omega$, then the distance

$$d_\Omega(P,Q) = \inf\{ L(\gamma) : \gamma \text{ is piecewise } C^1 \text{ curve from } P \text{ to } Q \}.$$

Lemma: Let $\Omega \subseteq \mathbb{C}^n$ be a bdd domain.

Then $g_{i\bar{j}}(z) > 0$. ← positive definite

Pf: Fix $z = P \in \Omega$. Recall in the last lecture, we have

① $\exists$ an o.n.b $\{\varphi_K\}_{K=0}^{\infty}$ of $A^2(\Omega)$ s.t

- $\varphi_0(P) = 0$
- $\varphi_1(P) = 0$, $\dfrac{\partial \varphi_1}{\partial z_1}(P) > 1$,
- $\varphi_2(P) = 0$, $\dfrac{\partial \varphi_2}{\partial z_1}(P) = 0$, $\dfrac{\partial \varphi_2}{\partial z_2}(P) = 0$

$\vdots$

- $\varphi_n(p)=0,\ \frac{\partial \varphi_n}{\partial z_1}(p)=0, \cdots, \frac{\partial \varphi_n}{\partial z_{n-1}}(p)=0,\ \frac{\partial \varphi_n}{\partial z_n}(p)>0.$

- For $k \geq n+1$, $D^\alpha \varphi_k(p)=0$ for $|\alpha|=0$ or 1.

② we will also need the following fact.

Fact: for $\forall$ o.n.b $\{\varphi_k\}_{k=0}^{\infty}$ of $A^2(\Omega)$

$$K(z,\bar{\xi}) = \sum_{k=0}^{\infty} \varphi_k(z)\overline{\varphi_k(\xi)}$$

converges uniformly for $(z,\xi) \in K_1 \times K_2$ with $K_1 \subset\subset \Omega$, $K_2 \subset\subset \Omega$.

Now we compute $g_{i\bar{j}}^{(p)} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z,\bar{z})\Big|_p$

$$= \frac{1}{K}\frac{\partial^2 K}{\partial z_i \partial \bar{z}_j}\Big|_p - \frac{1}{K^2}\frac{\partial K}{\partial z_i}\frac{\partial K}{\partial \bar{z}_j}\Big|_p$$

Note $\cdot$ $K(p,\bar{p}) = \varphi_0(p)\overline{\varphi_0(p)} > 0$

$$\left.\frac{\partial K}{\partial z_i}\right|_p = \sum_{l=0}^{\infty} \left(\frac{\partial \phi_l}{\partial z_i} \overline{\phi_l} \right) \Big|_p$$

$$= \left.\frac{\partial \phi_0}{\partial z_i}\right|_p \overline{\phi_0(p)}$$

Likewise, $\left.\frac{\partial K}{\partial \bar{z}_j}\right|_p = \phi_0(p) \left.\frac{\partial \overline{\phi_0}}{\partial \bar{z}_j}\right|_p$

$$\Rightarrow \quad \left.\frac{1}{K^2} \frac{\partial K}{\partial z_i} \frac{\partial K}{\partial \bar{z}_j}\right|_p = \frac{1}{|\phi_0(p)|^2} \left.\frac{\partial \phi_0}{\partial z_i}\right|_p \left.\frac{\partial \overline{\phi_0}}{\partial \bar{z}_j}\right|_p$$

$$\left.\frac{\partial^2 K}{\partial z_i \partial \bar{z}_j}\right|_p = \sum_{l=0}^{n} \left.\frac{\partial \phi_l}{\partial z_i} \frac{\partial \overline{\phi_l}}{\partial \bar{z}_j}\right|_p$$

$\Rightarrow$

$$g_{i\bar{j}}(p) = \frac{1}{|\phi_0(p)|^2} \sum_{l=1}^{n} \frac{\partial \phi_l}{\partial z_i} \overline{\left(\frac{\partial \phi_l}{\partial z_j} \right)} \Big|_p$$

$\Rightarrow$

$$\left(g_{i\bar{j}}(p) \right)_{1 \le i, j \le n} = \frac{1}{|\phi_0(p)|^2} A \bar{A}^t ,$$

where $A = \left(\frac{\partial \phi_l}{\partial z_i} \right)_{1 \le i, l \le n}$

Note $\det A \neq 0$ (Why?).

$\Rightarrow (g_{i\bar{j}}(p)) > 0$.

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proposition: Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ be domain and $f=(f_1,\cdots,f_n): \Omega_1 \to \Omega_2$ be a biholomorphism. Then f induces an isometry of Bergman metrics: in the sense that

$$\left((g_{\Omega_1})_{i\bar{j}}(p)\right) = (Jf)\big|_p \left((g_{\Omega_2})_{k\bar{l}}(f(p))\right) \overline{(Jf)}^t\big|_p$$

Bergman metric of Ω_1

Bergman metric of Ω_2

where $Jf = \left(\frac{\partial f_k}{\partial z_i}\right)_{1\leq i,k\leq n}$

$\Leftrightarrow |\xi|_{B,p} = |(Jf)\xi|_{B,f(p)}$.

Consequently, f preserves the Bergman distances:

$$d_{\Omega_2}(f(P), f(Q)) = d_{\Omega_1}(P, Q), \ \forall P, Q \in \Omega_1.$$

Pf: Read P11 in Krantz's book.

Some facts about the Bergman metric:

- Let Ω be a bdd domain.

  When (Ω, d_Ω) is complete?

  Bergman distance

What is known:

(Ω, d_Ω) complete $\Rightarrow$ Ω pseudoconvex

But the following Q is still open:

Q: Let Ω pseudoconvex, is (Ω, d_Ω) complete?

It's only known to be true if $\partial\Omega$ has some regularity.
But still open in general.

Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ be two bdd domain and $f: \Omega_1 \to \Omega_2$. Recall we have show

- $K_{\Omega_1}(z,\bar{z}) = K_{\Omega_2}(f(z), \overline{f(z)})\, |\det Jf(z)|^2$

- $\left((g_{\Omega_1})_{i\bar{j}}\right) = Jf\left(\left((g_{\Omega_2})_{i\bar{j}}\right)\right) \overline{Jf}^t$

$$\Rightarrow \det\left((g_{\Omega_1})_{i\bar{j}}\right) = \det\left((g_{\Omega_2})_{i\bar{j}}\right) |\det Jf(z)|^2$$

Hence
$$\frac{K_{\Omega_1}(z,\bar{z})}{\det\left((g_{\Omega_1})_{i\bar{j}}\right)} = \frac{K_{\Omega_2}(f(z), \overline{f(z)})}{\det\left((g_{\Omega_2})_{i\bar{j}}\right)}, \quad \forall z \in \Omega_1$$

Therefore, '$\dfrac{K_\Omega(z,\bar{z})}{\det\left((g_\Omega)_{i\bar{j}}\right)}$' is biholomorphic invariant.

Bergman introduced the following definition.

Def^n: Let $\Omega \subseteq \mathbb{C}^n$ be a bdd domain.

write K_Ω for its Bergman kernel

and $(g_\Omega)_{i\bar{j}}$ for its Bergman metric.

Define
$$B_\Omega(z,\bar{z}) = \frac{\det(g_\Omega)_{i\bar{j}}}{K_\Omega}$$

Remark: B_Ω is often called the Bergman invariant function of Ω.

Facts: (1) If $\Omega_1 \xrightarrow{f} \Omega_2$ is a biholomorphism, then $B_{\Omega_1}(z,\bar{z}) = B_{\Omega_2}(f(z), \overline{f(z)})$.

(2) Let Ω be a bdd domain. Write

$\mathrm{Aut}(\Omega) = \{f : f$ is a self-biholomorphism of $\Omega\}$.

we say Ω is homogeneous if $\mathrm{Aut}(\Omega)$ acts transitively on Ω. We have

if Ω is homogeneous $\Rightarrow$ $B_\Omega(z,\bar{z})$ is constant on Ω.

E.g (1). Let $P = \Delta^n$ be the unit polydisc in $\mathbb{C}^n$.

prove P is homogeneous. one can compute

$$B_P(z,\bar{z}) \equiv 2^n \pi^n$$

(2) Let $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball

in $\mathbb{C}^n$. Indeed $\mathbb{B}^n$ is also homogeneous. one

can compute $B_{\mathbb{B}^n}(z,\bar{z}) \equiv \dfrac{(n+1)^n \pi^n}{n!}$

Consequently, $\mathbb{B}^n$ and Δ^n are not biholomorphic

Recall

Let $(\Omega, g_{i\bar{j}})$ be a Hermitian metric with $g_{i\bar{j}}(z) \in C^\infty(\Omega)$

We say $g = (g_{i\bar{j}})$ is Kähler if at every pt $p \in \Omega$, locally $\exists$ smooth ϕ near p.

s.t $g_{i\bar{j}} = \dfrac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}$. near p.

Then the Bergman metric $g_{i\bar{j}}$ of Ω is Kähler.

When $g_{i\bar{j}}$ is Kähler, the Ricci tensor

$$Ric_{i\bar{j}} = - \frac{\partial^2 \log(\det(g_{i\bar{j}}))}{\partial z_i \partial \bar{z}_j}$$

Def^n: Let $g_{i\bar{j}}$ be a Kähler metric on Ω.

We say it is Einstein if

$Ric_{i\bar{j}} = \lambda g_{i\bar{j}}$ for some $\lambda \in \mathbb{R}$.

In this case, we say $(\Omega, g_{i\bar{j}})$ is Kähler-Einstein.

λ: is called the Ricci constant

Proposition: Let Ω be a bdd homogeneous domain. Then its Bergman metric $(g_{i\bar{j}})$ is Kähler-Einstein with $\lambda = -1$.

Pf: Recall since Ω is homogeneous,

$$B_\Omega = \frac{\det(g_{i\bar{j}})}{K_\Omega} \equiv c > 0$$

$$\Rightarrow \quad \det(g_{i\bar{j}}) = c K_\Omega$$

Apply $\partial_i \bar{\partial}_j \log \Rightarrow$

$$-Ric_{i\bar{j}} = g_{i\bar{j}}$$

or $$Ric_{i\bar{j}} = -g_{i\bar{j}} \qquad \text{四}$$

Yau's conjecture: (which is still open)

Let Ω be a bdd pseudoconvex domain.

The Bergman metric of Ω is Kähler-Einstein

$\Longleftrightarrow$ Ω is homogeneous.

Two related results

- (Cheng-Yau, Mok-Yau). Let $\Omega \subseteq \mathbb{C}^n$ be a bdd domain. Then Ω admits a complete Kähler-Einstein metric. $\Longleftrightarrow$ Ω is pseudoconvex.

- If $\Omega \subseteq \mathbb{C}^n$ is a bdd homogeneous domain, then Ω is pseudoconvex.