Recall let $\Omega \subseteq \mathbb{C}^n$ be a bdd domain.

Let $\{\varphi_k\}$ be an o.n.b in $A^2(\Omega)$.

The Bergman kernel $K = K_\Omega$ is defined as

$$K(z,\bar{\zeta}) = \sum_{k=0}^{\infty} \varphi_k(z)\overline{\varphi_k(\zeta)}, \quad z, \zeta \in \Omega.$$

Note: $\quad K(z,\bar{\zeta}) = \overline{K(\zeta,\bar{z})}$

Thm. For $\forall f \in A^2(\Omega)$, we have

$$f(z) = \int_\Omega K(z,\bar{\zeta}) f(\zeta)\, dV(\zeta), \quad \forall z \in \Omega.$$

Pf. Write $\quad f(z) = \sum_{k=0}^{\infty} a_k \varphi_k(z)$, where

$$a_k = \int_\Omega f(\zeta)\overline{\varphi_k(\zeta)}\, dV(\zeta)$$

It suffices to show

$$\lim_{m\to\infty} \left\{ \int_\Omega f(\zeta) K(z,\bar{\zeta})\, dV(\zeta) - \sum_{k=0}^{m} a_k \varphi_k(z) \right\} = 0$$

By Schwarz inequality and Fatou's lemma,

$$\left| \int_\Omega f(\zeta) K(z,\bar{\zeta})\, dV(\zeta) - \sum_{k=0}^{m} a_k \varphi_k(z) \right|$$

$$= \left| \int_\Omega f(\zeta) \sum_{k=m+1}^{\infty} \varphi_k(z) \overline{\varphi_k(\zeta)}\, dV(\zeta) \right|$$

$$\leq \|f\| \lim_{N\to\infty} \left\{ \int_\Omega \left| \sum_{k=m+1}^{N} \varphi_k(z) \overline{\varphi_k(\zeta)} \right|^2 \right\}^{\frac{1}{2}}$$

$$= \|f\| \left(\sum_{k=m+1}^{\infty} |\varphi_k(z)|^2 \right)^{\frac{1}{2}} \to 0 \text{ as } m \to \infty .$$

Cor. For $\forall g \in L^2(\Omega)$, set

$$(Pg)(z) = \int_\Omega K(z,\bar{\zeta})\, g(\zeta)\, dV(\zeta).$$

Then P is the orthogonal projection from $L^2(\Omega)$ to $A^2(\Omega)$.

<u>Pf</u>: Recall $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$. Therefore, each $g \in L^2(\Omega)$ can be decomposed as

$$g = g_1 + g_2, \text{ where}$$

$g_1 \in A^2(\Omega)$, $g_2 \perp A^2(\Omega)$.

Note: $Pg = Pg_1 + Pg_2 = g_1 + Pg_2$,

Furthermore, since $K(\cdot, \bar{z}) \in A^2(\Omega)$ for fixed $z \in \Omega$. $\Rightarrow$

$$Pg_2(z) = \int_\Omega K(z, \bar{\xi})\, g_2(\xi)\, dV(\xi)$$

$$= \int_\Omega \overline{K(\xi, \bar{z})}\, g_2(\xi)\, dV(\xi) = 0$$

Hence $Pg = g_1$.

Remark: By Functional analysis, for a projection operator $P^2 = P$, it is an orthogonal projection $\Longleftrightarrow$ P is self-adjoint

$\left(\Longleftrightarrow K(z, \bar{\xi}) = \overline{K(\xi, \bar{z})}\right)$

Proposition: Let $K'(z,\bar{\zeta})$ be a function in $\Omega\times\Omega$ satisfying

$$(*)\begin{cases} \overline{K'(z,\bar{\zeta})} \in A^2(\Omega),\ \forall \text{ fixed } z\in\Omega \\ f(z)=\int_\Omega f(\zeta)K'(z,\bar{\zeta})\,dV,\ \forall f\in A^2(\Omega) \end{cases}$$

Then $K'(z,\bar{\zeta}) = K(z,\bar{\zeta})$

Pf: Consider the bdd functional on $A^2(z)$: fixing $z\in\Omega$,

$$f\in A^2(\Omega) \xrightarrow{L} f(z)$$

By Riesz $\Rightarrow$

$\exists !\ g \in A^2(\Omega)$ s.t

$$f(z) = L(f) = \langle f, g \rangle$$
$$= \int_\Omega f(\xi) \overline{g(\xi)}\, dV$$

By $(*)$ and property of $K(z, \bar{\xi}) \Rightarrow$

$$\overline{K'(z, \bar{\xi})} = \overline{K(z, \bar{\xi})} = g(\xi)$$

$\Rightarrow \quad K'(z, \bar{\xi}) = K(z, \bar{\xi})$

Remark: The above proposition shows:
The definition of $K(z, \bar{\xi}) = \sum_{k=1}^{\infty} \varphi_k(z) \overline{\varphi_k(\xi)}$ does NOT depends on the choice of o.n.b $\{\varphi_k\}$.

Proposition: Let $\Omega_1 \subseteq \mathbb{C}^n$, $\Omega_2 \subseteq \mathbb{C}^m$ be the bdd domains. Let $\Omega = \Omega_1 \times \Omega_2$.

Write K_1, K_2, K for the Bergman kernels of $\Omega_1, \Omega_2, \Omega$, respectively. Then $K = K_1 K_2$.

Pf: Write $z' = (z_1, \cdots, z_n) \in \mathbb{C}^n$ for the coordinates of $\mathbb{C}^n$, $z'' = (z_{n+1}, \cdots, z_{n+m})$ for the coordinates of $\mathbb{C}^m$. Likewise for $\zeta', t' \in \mathbb{C}^n$, $\zeta'', t'' \in \mathbb{C}^m$. Write $z = (z', z'')$ and $\zeta = (\zeta', \zeta'')$, $t = (t', t'')$

Write $g(z, \bar{\zeta}) = K_1(z', \bar{\zeta}') K_2(z'', \bar{\zeta}'')$

We need to show $g(z, \bar{\zeta}) = K(z, \bar{\zeta})$

Note by the reproducing property of $K \Rightarrow$

$$g(z,\bar{\xi}) = \int_{\Omega} g(t,\bar{\xi})\, K(z,\bar{t})\, dV(t)$$

$$= \int_{\Omega_1\times\Omega_2} K_1(t',\bar{\xi}')\, K_2(t'',\bar{\xi}'')\, K(z,\bar{t})\, dV(t)$$

E.x: Use the reproducing property of K_1, K_2 to show

$$g(z,\bar{\xi}) = K(z,\bar{\xi})$$

Thm: (transformation Law of Bergman Kernel)

Let Ω and $\tilde{\Omega}$ be two bdd domains in $\mathbb{C}^n$, and $w = f(z)$ is a biholomorphism from Ω to $\tilde{\Omega}$. Write K and $\tilde{K}$ for the Bergman kernel of Ω and $\tilde{\Omega}$, respectively. $\Rightarrow$

$$K(z, \bar{\xi}) = \tilde{K}(w, \bar{\eta})\, Jf(z)\, \overline{Jf(\xi)}$$

Here $w = f(z)$, $\eta = f(\xi)$

$$Jf = \det\left(\frac{\partial f_i}{\partial z_j}\right)$$

$f = (f_1, \cdots, f_n)$, $z = (z_1, \cdots, z_n)$

$$\left(\xrightarrow{\text{let } \xi = z} K(z, \bar{z}) = \tilde{K}(f(z), \overline{f(z)})\, |Jf(z)|^2 \right)$$

Pf: Let $\{\varphi_K(w)\}$ be an o.n.b of $A^2(\Omega)$

i.e

$$\int_{\tilde{\Omega}} \varphi_K(w)\overline{\varphi_L(w)}\,dV(w) = \delta_{KL} = \begin{cases} 0 & \text{if } K \neq L \\ 1 & \text{if } K = L \end{cases}$$

$$\Rightarrow \int_\Omega \varphi_K(f(z))\overline{\varphi_L(f(z))}\,|Jf|^2 dV(z) = \delta_{KL}$$

$\Rightarrow \{\varphi_K(f(z))Jf(z)\}$ is an o.n.s.

Similarly, one can show since $\{\varphi_K(w)\}$ is complete $\Rightarrow \{\varphi_K(f(z))Jf(z)\}$ is complete

E.X.

Hence $\{\varphi_K(f(z))Jf(z)\}$ is an o.n.b.

$\Rightarrow$

$$K(z,\bar{\zeta}) = \sum_{k=1}^{\infty} \left(\varphi_k(f(z))\, Jf(z)\right) \left(\overline{\varphi_k(f(\zeta))\, Jf(\zeta)}\right)$$

$$= \tilde{K}(w,y)\, Jf(z)\, \overline{Jf(\zeta)}$$