In this note, we prove

**Theorem (Bergman):** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \). Then \( A^2(\Omega) \) has countable o.n.b.

**Proof:** WLOG, we assume \( 0 \in \Omega \). We will proceed in 3 steps.

**Step 1:** We first fix an order on multi-indices.
Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \beta = (\beta_1, \ldots, \beta_n) \). We say \( \beta < \alpha \) if either (a) or (b) holds:

(a): \( |\beta| < |\alpha| \)

(b): \( |\beta| = |\alpha| \) and \( \exists \) some \( 1 \leq k \leq n \) s.t.
\( \alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}, \alpha_k > \beta_k \).

Next fix each multi-index \( \alpha \),

\[ E^\alpha = \{ f \in A^2(\Omega) : D^\beta f(0) = 0 \text { for } \beta < \alpha \} \]

**Note:** \( E^\alpha \neq \emptyset \). Indeed, \( \frac{\alpha^\alpha}{\alpha!} \in E^\alpha \).
Consider the following minimizing \( \mathbf{Q} \).

\[ \mathbf{Q}: \text{Fix } \alpha. \text{ Does } \exists h \in E^\alpha \text{ s.t. } \|h\|^2 = A \leq \inf \|f\|^2: f \in E^\alpha. \]

\[ \text{A: Yes.} \]

\[ \textbf{Pf.}: \text{First } \exists \text{ a sequence } \{f_j\} \text{ s.t. } \|f_j\|^2 \to A \]
\[ \Rightarrow \|f_j\| \text{ is bdd} \]

By the previous lecture, we have on any compact subset \( K \subseteq \mathbb{R} \), we have \( \exists M_k > 0 \)
\[ |f_j(z)| \leq M_k \|f_j\| \]

\[ \Rightarrow f_j \text{ is locally bdd on } K \]

By Montel's thm, \( \exists \) a subsequence \( \{f_{j_i}\} \) s.t.
\[ f_{j_i} \to h \in H(\mathbb{C}) \text{, normally} \]

\[ \Rightarrow D^n h(0) = 1, \quad D^B h(0) = 0, \text{ if } B < \alpha. \]

For every \( \mathbf{K} \subset \subset \mathbb{R} \), \( f_{j_i} \to h \text{ uniformly on } K \)

\[ \int_K |h|^2dv = \lim_{n \to \infty} \int_K |f_{j_i}|^2dv \leq \lim_{n \to \infty} \int |f_{j_i}|^2dv \leq A \]
\[ \Rightarrow \int_\Omega 1 \, h^2 \, dv \leq A \]

Hence \[ \int_{\mathbb{R}^2} h \in \mathbb{E}^2 \]
\[ 11h11^2 \leq A \]

By the defn of A, \( \Rightarrow 11h11^2 \geq A \)

Therefore \( 11h11^2 = A \).

In addition, since \( D^2 h(0) = I \), \( \Rightarrow h \neq 0 \)

\( \Rightarrow A > 0 \). We will write \( h_\alpha \) for \( h \), as it depends on \( \alpha \).

Step 2.

Lemma: \( \{ h_\alpha \}_{\alpha \in (0^\circ, \pi^\circ)}^n \) is an orthogonal system

PF: We will first prove the following claim.
Claim: For each $\alpha$, if $g \in \mathcal{A}^2(\alpha)$ satisfies
\[ D^\beta g(t) = 0, \quad \forall \beta \leq \alpha, \text{ then } \langle h_\alpha, g \rangle = 0. \]

Proof: WLOG, assume $g \neq 0$. Note for $\forall c \in C$,
\[ D^\beta (h_\alpha + c g)(t) = 0, \quad \text{if } \beta < \alpha \]
\[ D^\alpha (h_\alpha + c g)(t) = 1. \]

\[ \Rightarrow h_\alpha + c g \in \mathcal{E}^2 \Rightarrow \| h_\alpha + c g \|^2 \geq A. \]

On the other hand,
\[ \| h_\alpha + c g \|^2 = \| h_\alpha \|^2 + |c|^2 \| g \|^2 + \langle c g, h_\alpha \rangle + \bar{c} \langle h_\alpha, g \rangle. \]

Pick $c = -\frac{\langle h_\alpha, g \rangle}{\| g \|^2}$ \Rightarrow
\[ \| h_\alpha + c g \|^2 = \| h_\alpha \|^2 - \frac{\| h_\alpha \|^2}{\| g \|^2} \geq A = \| h_\alpha \|^2. \]

We must have $\langle h_\alpha, g \rangle = 0$.

Corollary: $\exists \ h_\alpha \in \mathcal{E}^2 \text{ st } \| h_\alpha \|^2 = A$.

Proof: E.x. Hint: Suppose $h_\alpha, \ h_{\alpha} \in \mathcal{E}^2 \text{ st }$
\[ \| h_\alpha \|^2 = \| h_{\alpha} \|^2 = A. \]
prove $\langle h_\lambda - \hat{h}_\lambda, h_\lambda \rangle = \langle h_\lambda - \hat{h}_\lambda, h_\lambda \rangle = 0$

$\Rightarrow \langle h_\lambda - \hat{h}_\lambda, h_\lambda - \hat{h}_\lambda \rangle = 0$.

Corollary: If $\alpha \neq \beta$, then $\langle h_\alpha, h_\beta \rangle = 0$.

Pf: It follows from the lemma.

Hence $\mathcal{F}h_\lambda$ is an orthogonal system.

Let $\psi_\lambda = \frac{h_\lambda}{\|h_\lambda\|}$. Then $\mathcal{F}\psi_\lambda \in (\mathbb{Z}^2)^n$ is an orthonormal system. It remains to show

Lemma: $\mathcal{F}\psi_\lambda$ is complete. (That is, for $g \in H^2(\mathbb{C})$,

if $g \perp \psi_\lambda$ for all $\lambda$, then $g = 0$)

Pf: We have assigned $(\mathbb{Z}^2)^n$ an order. We list

$(\mathbb{Z}^2)^n$ as:

$\begin{array}{c}
\langle \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots \\
\suppose g \perp \psi_{\lambda_k}, i.e. \langle g, \psi_{\lambda_k} \rangle \text{ for every } k \geq 1.
\end{array}$
we will show \( g = 0 \).

1. Write \( c_i = (D^2 g)(0) 11h_{11} \) and

\[
g = c_i \varphi_{a1} + (g - c_i \varphi_{a1})
\]

Then \( D^1 g \varphi_{a1} = 0 \). By the above claim:

\[
<g - c_i \varphi_{a1}, \varphi_{a1}> = 0 \implies
\]

\[
0 = <g, \varphi_{a1}> = <c_i \varphi_{a1}, \varphi_{a1}> = c_i
\]

\[\implies c_i = 0\]

2. Write \( c_2 = D^2 g 11h_{21} \)

\[
= c_2 \varphi_{a2} + (g - c_2 \varphi_{a2})
\]

Note \( D^1 (g - c_2 \varphi_{a2}) = 0 \) (why?)

\[
D^2 (g - c_2 \varphi_{a2}) = 0
\]

\[\implies <g - c_2 \varphi_{a2}, \varphi_{a2}> = 0\]

\[\implies 0 = <g, \varphi_{a2}> = c_2\]
In this way, we can show \( u \not\in H_\infty \) for \( u \in H \).

Since \( g \in H_\infty \) \( \Rightarrow \) \( g \equiv 0 \).

**Cor.** Let \( \Omega \) be a bounded domain. Then

\[
K(z, \bar{z}) > 0 \quad \text{for all } z \in \Omega.
\]

*the Bergman kernel of \( \Omega \).*

**Pf.** E.X.

Next we study the o.n.d of \( A^2(\Omega) \) for a special class of domains \( \Omega \) (Reinhardt domain).

**Defn.** Let \( \Omega \subseteq \mathbb{C}^n \) be a domain.

We say \( \Omega \) is Reinhardt if for

\[
\forall (z_1, \ldots, z_n) \in \Omega \text{ and } \forall (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n
\]

we have \( e^{i\theta_1 z_1}, \ldots, e^{i\theta_n z_n} \in \Omega. \)
**Proposition.** Let $\Omega \subseteq \mathbb{C}^n$ be a Reinhardt domain containing $0 \in \mathbb{C}^n$. Let $f \in H(\Omega)$. Then $f$ can be written as a power series in $\Omega$:

$$f(z) = \sum_{\alpha} a_{\alpha} z^\alpha$$

with normal convergence in $\Omega$.

**Proof.** Read P35 in Hörmander's book.

**Theorem.** Let $\Omega \subseteq \mathbb{C}^n$ be a balanced Reinhardt domain and $0 \in \Omega$. Then

$$\mathcal{F} \sum_{\alpha} \frac{z^\alpha}{1 + 2^\alpha} (\mathbb{Z}^n)^n$$

is an o.n.b of $A^2(\Omega)$. 
pf: Fix $\alpha \in (\mathbb{Z}^{2n})^n$. Recall

$$E^\alpha = \{ f \in A^2(\mathbb{R}) : D^\alpha f(0) = 1, D^\beta f(0) = 0 \text{ for } \beta < \alpha \}$$

It suffices to show

\[ \text{Claim: } \| \frac{\partial^\alpha}{\partial^\alpha} f \|^2 = \int_{\mathbb{R}^n} \sum_{\beta > \alpha} \| a_\beta \|_2^2 \| f \|^2 \text{ for } \beta > \alpha \]

Let $f \in E^\alpha$. Then by the previous proposition,

$$f = \sum_{\beta > \alpha} a_\beta z^\beta$$

$$\Rightarrow \| f \|^2 = \int_{\mathbb{R}^n} f(z) \overline{f(z)} \, dz$$

$$\geq \| \frac{\partial^\alpha}{\partial^\alpha} f \|^2 + \sum_{\beta > \alpha} \| a_\beta \|_2^2 \| f \|^2$$

\[ \square \]

Hence the claim holds. \[ \square \]
Let \( IB^n = \{ z \in C^n : |z| < 1 \} \). Then by Thm.
\[
\left\{ \frac{z^d}{|z|^d} \right\} \text{ is an o.n.b of } A^2(\Lambda^n) \]

as \( \Lambda^n \) is Reinhardt and \( \Theta \in \Lambda^n \). Using this, we can prove the Bergman kernel of \( \Lambda^n \).

Thm. Write \( K_{\Lambda^n} \) for the Bergman kernel of \( \Lambda^n \). Then
\[
K_{\Lambda^n}(z, \bar{z}) = \frac{n!}{\pi^n} \frac{1}{(1 - \langle z, \bar{z} \rangle)^{n+1}}
\]
Here \( \langle z, \bar{z} \rangle = z \cdot \bar{z} \)

Consequently,
\[
K_{\Lambda^n}(z, \bar{z}) = \frac{n!}{\pi^n} \frac{1}{(1 - |z|^2)^{n+1}}
\]
PF: By the defn of \( \text{KiB}^n \),

Letting \( \psi_2 = \frac{z^2}{11z^{2\cdot 11}} \)

\[
\text{KiB}^n = \sum_{\alpha \in (\mathbb{Z}^+) \cdot 11} \psi_2 (\alpha) \overline{\psi_2 (\alpha)}
\]

We need to compute \( \| \text{KiB}^n \| \)

\[
\| \text{KiB}^n \|^2 = \int_{\text{KiB}^n} z^2 \overline{z^2} \, dV
\]

E.x: prove the above equals to

\[
\frac{\pi^n 2!}{(n+1d1)!}
\]

Hint: use polar coordinates for each \( z^j \)
Then

\[ k_{1B}^{n}(z, \bar{z}) = \sum_{\alpha \in [2^{\infty}]} \frac{(n+|\alpha|)!}{\pi^{n} \alpha!} 2^{n-|\alpha|} \bar{z}^{\alpha} \]

\[ = \frac{1}{\pi^{n}} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \sum_{|\alpha|=k} \frac{k!}{\alpha!} 2^{n-|\alpha|} \bar{z}^{\alpha} \]

\[ = \frac{1}{\pi^{n}} \sum_{k=0}^{\infty} (k+1) \ldots (n+k) <z, \bar{z}>^{k} \]

\[ = \frac{n!}{\pi^{n}} \frac{1}{(1-<z, \bar{z}>)^{n+1}} \]

Note: \( k_{1B}^{n}(z, \bar{z}) \to 0 \) as \( z \to 0 \) in \( B^{n} \).
Remark: In particular, we have the Bergman kernel of the unit disk (i.e., $n=1$ in the above)

$$K_{\Delta} = \frac{1}{\pi} \frac{1}{(1 - \langle z, \overline{z} \rangle)^2}$$