In this note, we prove

Thm: (Bergman) Let Ω be a bdd domain in $\mathbb{C}^n$.

Then $A^2(\Omega)$ has countable o.n.b.

Pf: WLOG, we assume $0 \in \Omega$.

We will proceed in 3 steps.

Step 1: We first fix an order on multiindices:

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$. We say $\beta < \alpha$ if either (a) or (b) holds:

(a): $|\beta| < |\alpha|$

(b): $|\beta| = |\alpha|$. and $\exists$ some $1 \le k \le n$ s.t

$\alpha_1 = \beta_1, \cdots, \alpha_{k-1} = \beta_{k-1}, \alpha_k > \beta_k$.

Next for each multiindex α,

$$E^\alpha = \{ f \in A^2(\Omega) : D^\alpha f(0) = 1, D^\beta f(0) = 0 \text{ for } \beta < \alpha \}.$$

Note: $E^\alpha \neq \phi$. Indeed. $\frac{z^\alpha}{\alpha!} \in E^\alpha$.

Consider the following minimizing Q.

Q: Fix α. Does $\exists\, h \in E^{\alpha}$ s.t

$$\|h\|^2 = A \triangleq \inf\{\|f\|^2 : f \in E^{\alpha}\}.$$

A: Yes.

Pf: First $\exists$ a sequence $\{f_j\}$ s.t $\|f_j\|^2 \to A$

$$\Rightarrow \{\|f_j\|\} \text{ is bdd}$$

By the previous Lecture, we have on any compact subset $K \subseteq \Omega$, we have $\exists\, M_K > 0$

$$|f_j(z)| \leq M_K \|f_j\|$$

$\Rightarrow$ f_j is locally bdd on K

By Montel's thm, $\exists$ a subsequence $\{f_{j_i}\}$ s.t

$$f_{j_i} \to h \in H(\Omega) \text{ . normally}$$

$\Rightarrow$ $D^{\alpha}h(0) = 1$, $D^{\beta}h(0) = 0$. if $\beta < \alpha$.

For $\forall\, K \subset\subset \Omega$, $f_{j_i} \to h$ uniformly on K

$$\int_K |h|^2 dV = \lim \int_K |f_{j_i}|^2 dV \leq \lim \int_{\Omega} |f_{j_i}|^2 dV \leq A$$

$\Rightarrow \int_\Omega |h|^2 dV \le A$

Hence $\begin{cases} h \in E^\alpha \\ \|h\|^2 \le A \end{cases}$

By the defn of A, $\Rightarrow \|h\|^2 \ge A$

Therefore $\|h\|^2 = A$.

In addition, since $D^\alpha h(0) = 1$, $\Rightarrow h \not\equiv 0$

$\Rightarrow A > 0$. We will write h_α for h, as it depends on α.

Step 2.

Lemma: $\{h_\alpha\}_{\alpha \in (\mathbb{Z}^{\ge 0})^n}$ is an orthogonal system

Pf: We will first prove the following claim.

Claim: For each α, if $g \in A^2(\Omega)$ satisfies $D^\beta g(0)=0$, $\forall \beta \le \alpha$, then $\langle h_\alpha, g\rangle = 0$.

Pf: WLOG, assume $g \not\equiv 0$. Note for $\forall c \in \mathbb{C}$,

$$\begin{cases} D^\beta (h_\alpha + cg)(0) = 0, & \text{if } \beta < \alpha \\ D^\alpha (h_\alpha + cg)(0) = 1. \end{cases}$$

$$\Rightarrow \quad h_\alpha + cg \in E^\alpha \Rightarrow \|h_\alpha + cg\|^2 \ge A$$

On the other hand,

$$\|h_\alpha + cg\|^2 = \|h_\alpha\|^2 + |c|^2\|g\|^2 + c\langle g, h_\alpha\rangle + \bar{c}\langle h_\alpha, g\rangle$$

pick $c = -\dfrac{\langle h_\alpha, g\rangle}{\|g\|^2}$ $\Rightarrow$

$$\|h_\alpha + cg\|^2 = \|h_\alpha\|^2 - \frac{|\langle h_\alpha, g\rangle|^2}{\|g\|^2} \ge A = \|h_\alpha\|^2$$

We must have $\langle h_\alpha, g\rangle = 0$.

Corollary: $\exists !\, h_\alpha \in E^\alpha$ s.t $\|h_\alpha\|^2 = A$.

Pf: E.x. Hint: suppose $h_\alpha, \hat{h}_\alpha \in E^\alpha$ s.t $\|h_\alpha\|^2 = \|\hat{h}_\alpha\|^2 = A$.

prove $\langle h_\alpha - \hat{h}_\alpha, h_\alpha \rangle = \langle h_\alpha - \hat{h}_\alpha, \hat{h}_\alpha \rangle = 0$

$\Rightarrow \langle h_\alpha - \hat{h}_\alpha, h_\alpha - \hat{h}_\alpha \rangle = 0$.

Corollary: If $\alpha \neq \beta$, then $\langle h_\alpha, h_\beta \rangle = 0$.

Pf: It follows from the Lemma.

Hence $\{h_\alpha\}$ is an orthogonal system.

Let $\varphi_\alpha = \dfrac{h_\alpha}{\|h_\alpha\|}$. Then $\{\varphi_\alpha\}_{\alpha \in (\mathbb{Z}^{\geq 0})^n}$ is an orthonormal system. It remains to show

Lemma: $\{\varphi_\alpha\}$ is complete. (That is, for $g \in H^2(\Omega)$, if $g \perp \varphi_\alpha$ for all α, then $g = 0$)

Pf: We have assigned $(\mathbb{Z}^{\geq 0})^n$ an order. We list $(\mathbb{Z}^{\geq 0})^n$ as:

$\alpha^1 < \alpha^2 < \alpha^3 < \cdots < \alpha^k < \cdots$

Suppose $g \perp \varphi_{\alpha^k}$, i.e. $\langle g, \varphi_{\alpha^k} \rangle$ for every $k \geq 1$.

We will show $g = 0$.

① Write $c_1 = (D^{\alpha^1} g)(0) \|h_{\alpha^1}\|$ and

$$g = c_1 \varphi_{\alpha^1} + (g - c_1 \varphi_{\alpha^1})$$

Then $D^{\alpha^1}(g - c_1 \varphi_{\alpha^1}) = 0$. By the above claim,

$\langle g - c_1 \varphi_{\alpha^1}, \varphi_{\alpha^1} \rangle = 0 \quad \Rightarrow$

$$0 = \langle g, \varphi_{\alpha^1} \rangle = \langle c_1 \varphi_{\alpha^1}, \varphi_{\alpha^1} \rangle = c_1$$

$\Rightarrow c_1 = 0$

② Write $c_2 = D^{\alpha^2} g \quad \|h_{\alpha^2}\|$

$$= c_2 \varphi_{\alpha^2} + (g - c_2 \varphi_{\alpha^2})$$

Note $\begin{cases} D^{\alpha^1}(g - c_2 \varphi_{\alpha^2}) = 0 \quad \text{(why?)} \\ D^{\alpha^2}(g - c_2 \varphi_{\alpha^2}) = 0 \end{cases}$

$\Rightarrow \langle g - c_2 \varphi_{\alpha^2}, \varphi_{\alpha^2} \rangle = 0$

$\Rightarrow \quad 0 = \langle g, \varphi_{\alpha^2} \rangle = c_2$

In this way, we can show $D^{\alpha} g(0) = 0 \quad \forall \alpha \in (\mathbb{Z}^{\geq 0})^n$

In this way, we can show $D^{\alpha} g(0) = 0$, $\forall \alpha \in (\mathbb{Z}^{+})^{n}$

Since $g \in H(\Omega) \Rightarrow g \equiv 0$. $\blacksquare$

Cor. Let Ω be a bounded domain. Then

$$K(z, \bar{z}) > 0 \text{ for all } z \in \Omega.$$

the Bergman Kernel of Ω.

Pf: Ex.

Next we study the o.n.b of $A^2(\Omega)$ for a special class of domains Ω (Reinhardt domain)

Def^n: Let $\Omega \subseteq \mathbb{C}^n$ be a domain.

We say Ω is Reinhardt if for

$\forall (z_1, \cdots, z_n) \in \Omega$ and $\forall (\theta_1, \cdots, \theta_n) \in \mathbb{R}^n$

we have $(e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n) \in \Omega$.

Proposition: Let $\Omega \subseteq \mathbb{C}^n$ be a Reinhardt domain containing $0 \in \mathbb{C}^n$. Let $f \in H(\Omega)$. Then f can be written as a power series in Ω:

$$f(z) = \sum_{\alpha} a_\alpha z^\alpha$$

with normal convergence in Ω.

Pf: Read P35 in Hörmander's book.

Thm: Let $\Omega \subseteq \mathbb{C}^n$ be a bdd Reinhardt domain and $0 \in \Omega$. Then

$\left\{ \frac{z^\alpha}{\|z^\alpha\|} \right\}_{\alpha \in (\mathbb{Z}^{\geq 0})^n}$ is an o.n.b of $A^2(\Omega)$.

Pf: Fix $\alpha \in (\mathbb{Z}^{\geq 0})^n$. Recall

$$E^\alpha = \{ f \in A^2(\Omega) : D^\alpha f(0) = 1,\ D^\beta f(0) = 0 \text{ for } \beta < \alpha \}$$

It suffices to show

Claim: $\left\| \frac{z^\alpha}{\alpha!} \right\|^2 = \inf \{ \|f\|^2 : f \in E^\alpha \}$.

Let $f \in E^\alpha$. Then by the previous proposition,

$$f = \frac{z^\alpha}{\alpha!} + \sum_{\beta > \alpha} a_\beta z^\beta$$

$$\Rightarrow \quad \|f\|^2 = \int_\Omega f(z) \overline{f(z)} \, dV$$

$$\geq \left\| \frac{z^\alpha}{\alpha!} \right\|^2 + \sum_{\beta > \alpha} \| a_\beta z^\beta \|^2$$

Ex

Hence the claim holds. ▨

Let $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$. Then by Thm, $\left\{\frac{z^\alpha}{\|z^\alpha\|}\right\}$ is an o.n.b of $A^2(\mathbb{B}^n)$ as $\mathbb{B}^n$ is Reinhardt and $0 \in \mathbb{B}^n$. Using this, we can prove the Bergman kernel of $\mathbb{B}^n$.

<u>Thm</u>. Write K_{B^n} for the Bergma kernel of $\mathbb{B}^n$. Then

$$K_{\mathbb{B}^n}(z, \bar{\xi}) = \frac{n!}{\pi^n} \frac{1}{(1 - \langle z, \xi \rangle)^{n+1}}$$

Here $\langle z, \xi \rangle = z \cdot \bar{\xi}^t$

Consequently,

$$K_{\mathbb{B}^n}(z, \bar{z}) = \frac{n!}{\pi^n} \frac{1}{(1 - |z|^2)^{n+1}}$$

Pf: By the defn of $K_{\mathbb{B}^n}$,

letting $\varphi_\alpha = \frac{z^\alpha}{\|z^\alpha\|}$

$$K_{\mathbb{B}^n} = \sum_{\alpha \in (\mathbb{Z}^{\geq 0})^n} \varphi_\alpha(z) \overline{\varphi_\alpha(\zeta)}$$

We need to compute $\|z^\alpha\|$

$$\|z^\alpha\|^2 = \int_{\mathbb{B}^n} z^\alpha \overline{z^\alpha} \, dV$$

E.x: prove the above equals to

$$\frac{\pi^n \alpha!}{(n+|\alpha|)!}$$

Hint: use polar coordinates for each z_j

Then

$$
\begin{aligned}
&K_{\mathbb{B}^n}(z,\bar{\xi}) \\
&= \sum_{\alpha\in(\mathbb{Z}^{\geq 0})^n} \frac{(n+|\alpha|)!}{\pi^n\,\alpha!}\, z^\alpha \bar{\xi}^\alpha \\
&= \frac{1}{\pi^n}\sum_{k=0}^{\infty}\frac{(n+k)!}{k!}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\, z^\alpha \bar{\xi}^\alpha \\
&= \frac{1}{\pi^n}\sum_{k=0}^{\infty}(k+1)\cdots(n+k)\,\langle z,\xi\rangle^k \\
&= \frac{n!}{\pi^n}\,\frac{1}{(1-\langle z,\xi\rangle)^{n+1}} \qquad \blacksquare
\end{aligned}
$$

Note: $K_{\mathbb{B}^n}(z,\bar{z}) \to 0$ as $z \to \partial\mathbb{B}^n$.

Remark: In particular, we have the Bergman kernel of the unit disk (i.e., $n=1$ in the above)

$$K_\Delta = \frac{1}{\pi} \frac{1}{(1-\langle z, \xi \rangle)^2}$$