We first recall from one complex variables:

Def^n: Let Ω be an open set in $\mathbb{C}$.

Let $z = x+iy$ be the complex variables

Recall

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Let $f \in C^1(\Omega)$. Then f is called holomorphic

if $\frac{\partial f}{\partial \bar{z}} = 0$.

$\Longleftrightarrow$ writing $f = u+iv$, $u = \mathrm{Re} f$, $v = \mathrm{Im} f$.

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Cauchy-Riemann eqn.

Notation: $f \in H(\Omega) \Leftrightarrow f$ holomorphic in Ω.

Complex Analyticity:

If f is holomorphic in Ω, then for $\forall a \in \Omega$, $\exists$ an small open disk D centered at a, such that f can be expanded as a convergent power series:

$$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n$$

Cauchy formula:

Let Ω be a bounded domain with C^1 smooth boundary. Assume $f \in C^1(\bar{\Omega})$ and

$f \in H(\Omega)$, then

$$f(a) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-a} dz$$

Riemann mapping Thm:

Let U be a non-empty simply connected open subset of $\mathbb{C}$ and $U \neq \mathbb{C}$. Then U is biholomorphic to the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

Several complex Variables

$$\mathbb{C}^n = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n \text{ copies}}$$

Denote the complex coordinates

in $\mathbb{C}^n$ by $z = (z_1, \cdots, z_n)$.

where $z_k = x_k + i y_k$

with $x_k = \operatorname{Re} z_k$, $y_k = \operatorname{Im} z_k$.

Define for $1 \le k \le n$

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k}\right)$$

$$\frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k}\right).$$

Def^n: Let Ω be an open set in $\mathbb{C}^n$.
A function $u \in C^1(\Omega)$ is said to be holomorphic if $\dfrac{\partial f}{\partial \bar{z}_k} = 0$ for all $1 \le k \le n$.

$\Leftrightarrow$ writing $f = u + iv$,

$$\begin{cases} \dfrac{\partial u}{\partial x_k} = \dfrac{\partial v}{\partial y_k}, & 1 \le k \le n. \end{cases}$$

$\llcorner \dfrac{\partial u}{\partial y_k} = -\dfrac{\partial v}{\partial x_k}$,

Notation: $f \in H(\Omega) \iff f$ is holomorphic in Ω.

Two important domains:

① Let $b = (b_1, \cdots, b_n) \in \mathbb{C}^n$,
$\rho = (\rho_1, \cdots, \rho_n) \in \mathbb{R}^n$, $\rho_k \geq 0$.

Define

$$P(b, \rho) = \{ z \in \mathbb{C}^n \mid |z_k - b_k| < \rho_k, \ \forall\, 1 \leq k \leq n \}$$

That is,

$$P(b, \rho) = \Delta(b_1, \rho_1) \times \Delta(b_2, \rho_2) \times \cdots \times \Delta(b_n, \rho_n)$$

Unit polydisc centered at 0

$$\Delta^n = \underbrace{\Delta \times \cdots \times \Delta}_{n \text{ copies}}.$$

② Let $b=(b_1, \dots, b_n) \in \mathbb{C}^n$, $r>0$.

Define the ball

$$B(b,r) = \{ z \in \mathbb{C}^n \mid \|z-b\|^2 = \sum_{k=1}^{n} |z_k - b_k|^2 < r^2 \}.$$

Unit ball centered at 0:

$$\mathbb{B}^n = \{ z \in \mathbb{C}^n \mid \|z\|^2 = \sum_{k=1}^{n} |z_k|^2 < 1 \}.$$

Compare one CV and SCV.

$\mathbb{C}$	$\mathbb{C}^n$
Let $f \in H(\Omega)$ be nonconstant The zeros $\{z \mid f(z)=0\}$ must be isolated.	A holomorphic function can never have isolated zeros. Moreover, zeros must propagate either to infinity or to the bdry of the domain where the function is defined.

No such domain exists.	Hartogs phenomenon: $\exists$ a domain Ω in $\mathbb{C}^n$ such that every $f \in H(\Omega)$ can be extended to a larger domain
Riemann mapping thm	Even Δ^n and $\mathbb{B}^n$ are NOT biholomorphic.
The complex plane $\mathbb{C}$ can never be biholomorphic to its proper subset.	$\exists$ a proper subset Ω of $\mathbb{C}^n$ such that there is a biholomorphic map from Ω to $\mathbb{C}^n$. Such domains are called Fatou-Bieberbach domains

Cauchy Integral formula on polydisc.

Thm 1: Let $P(a,\rho)$ be a polydisc in $\mathbb{C}^n$.

Let f be a continuous function $\overline{P(a,\rho)}$.

Assume f is holomorphic in each z_j when the other variables are fixed.

Then we have

$$u(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_1 - a_1| = \rho_1} \cdots \int_{|\xi_n - a_n| = \rho_n} \frac{f(\xi_1, \cdots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi_1 \cdots d\xi_n$$

Consequently, f is holomorphic in $P(a,\rho)$.

proof: use induction on n.

① $n=1$, just the classical formula in $\mathbb{C}$.

② Assume it is true for $n-1$. Now we prove for n.

Consider $f(\zeta_1, \cdots, \zeta_{n-1}, z_n)$ with fixed $\zeta' = (\zeta_1, \cdots, \zeta_n) \in \prod_{k=1}^{n-1} \partial\Delta(a_k, \rho_k)$.

Claim: $f(\zeta_1, \cdots, \zeta_{n-1}, z_n)$ is holomorphic in $\Delta(a_n, \rho_n)$.

pf of Claim: let $z' = (z_1, \cdots, z_{n-1}) \in \prod_{k=1}^{n-1} \Delta(a_k, \rho_k)$ fixed

Then $f(z', z_n)$ is holomorphic in $z_n \in \Delta(a_n, \rho_n)$

since $f \in C^0(\overline{P(a,\rho)})$

$\Rightarrow \lim_{z' \to \zeta'} f(z', z_n) = f(\zeta', z_n)$ uniformly on any compact $K \subset \Delta(a_n, \rho_n)$

$\Rightarrow f(\zeta', z_n)$ is holomorphic on $\Delta(a_n, \rho_n)$

By Claim + "Cauchy formula for $n=1$"

$$\Rightarrow f(\zeta', z_n) = \frac{1}{2\pi i} \int_{|\zeta_n - z_n| = \rho_n} \frac{f(\zeta_1, \cdots, \zeta_n)}{\zeta_n - z_n} d\zeta_n \quad (1)$$

By the Inductive hypothesis for

$$(z_1, \cdots, z_{n-1}) \to f(z_1, \cdots, z_{n-1}, z_n)$$

$$f(z_1, \cdots, z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{|\xi_1 - a_1| = \rho_1} \cdots \int_{|\xi_{n-1} - a_{n-1}| = \rho_{n-1}} \frac{f(\xi_1, \cdots, \xi_{n-1}, z_n)}{(\xi_1 - z_1) \cdots (\xi_{n-1} - z_{n-1})} d\xi_1 \cdots d\xi_{n-1}$$

By (1)

$$= \frac{1}{(2\pi i)^{n-1}} \int_{|\xi_1 - a_1| = \rho_1} \cdots \int_{|\xi_n - a_n| = \rho_n} \frac{f(\xi_1, \cdots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi_1 \cdots d\xi_n .$$

Then $f \in C^1(P(a, \rho))$ (why?)

$\Rightarrow$ f is holomorphic in $P(a, \rho)$.

Remark 1: The bdry $\partial P(a,P)$ of $P(a,P)$ consists of many parts:

e.g $\quad \partial\Delta(a_1,P_1)\times\Delta(a_2,P_2)\times\cdots\times\Delta(a_n,P_n)$

$$\partial\Delta(a_1,P_1)\times\partial\Delta(a_2,P_2)\times\Delta(a_3,P_3)\times\cdots\times\Delta(a_n,P_n)$$

In particular, the part

$$\partial\Delta(a_1,P_1)\times\partial\Delta(a_2,P_2)\times\cdots\times\partial\Delta(a_n,P_n)$$

is called the characteristic / shilov bdry of $P(a,P)$.

denoted by $\partial_0 P(a,P)$

Q: Let $f\in H(P(a,P))$ and $f\in C^0(\overline{P(a,P)})$

prove $\displaystyle\max_{\overline{P(a,P)}}|f| = \max_{\partial_0 P(a,P)}|f|$

Remark 2: Thm 1 indicates, if $f \in C^0(\overline{P(a,\rho)})$,

then

separately holomorphic $\Rightarrow$ jointly holomorphic.

Recall in general,

separately differentiable $\not\Rightarrow$ jointly differentiable

E.g

$$u(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

In this regard, Hartogs showed in the above Thm 1,

the assumption that $f \in C^0(\overline{P(a,\rho)})$ can be dropped.

Hartogs Thm: If $f(z_1, \cdots, z_n)$ is holomophic in each z_j when other variables are fixed $\Rightarrow f$ is holomorphic in $(z_1, \cdots, z_n)$

That is, separately holomorphic $\Rightarrow$ jointly holomorphic.

Analyticity

Thm 2: Let f be holomorphic in $P(a,\rho)$. Then f can be expanded as a power series

$$f(z)=\sum_{\alpha}\frac{(D^{\alpha}f)(a)}{\alpha!}(z-a)^{\alpha},$$

Pf: Pick $\lambda=(\lambda_1,\cdots,\lambda_m)$ s.t $\overline{P(a,\lambda)}\subseteq P(a,\rho)$.

Then by Thm 1, we have,

$$f(z)=\frac{1}{(2\pi i)^n}\int_{|\xi_1-a_1|=\lambda_1}\cdots\int_{|\xi_n-a_n|=\lambda_n}\frac{f(\xi)}{(\xi_1-z_1)\cdots(\xi_n-z_n)}d\xi_1\cdots d\xi_n \quad (2)$$

we differentiate both sides to get

for $\alpha=(\alpha_1,\cdots,\alpha_n)$,

$$(D^{\alpha}f)(z)=\frac{\alpha!}{(2\pi i)^n}\int_{|\xi_1-a_1|=\lambda_1}\cdots\int_{|\xi_n-a_n|=\lambda_n}\frac{f(\xi)}{\prod_{j=1}^{n}(\xi_j-z_j)^{\alpha_j+1}}d\xi_1\cdots d\xi_n \quad (3)$$

Pick $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{R}^n$, s.t $\overline{P(a,\mu)} \subset P(a,\lambda)$.

Then when $z \in P(a,\mu)$ and $|\xi_j - a_j| = \lambda_j$, we have

$$\frac{1}{\prod_{j=1}^{n} (\xi_j - z_j)} = \sum_{\alpha_1, \cdots, \alpha_n = 0}^{\infty} \frac{(z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha_n + 1}}{(\xi_1 - a_1)^{\alpha_1 + 1} \cdots (\xi_n - a_n)^{\alpha_n + 1}} \quad (4)$$

and the series is uniformly conv. for $z \in P(a,\mu)$.

Multiply both sides of (4) by $f(\xi)$, integrate

w.r.t ξ on $\{|\xi_j - a_j| = \lambda_j : j = 1, \cdots, n\}$.

Then using (2) and (3), we get

$$f(z) = \sum_{\alpha} (z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha_n} \frac{1}{(2\pi i)^n} \int \cdots \int_{|\xi_1 - a_1| = \lambda_1, \, |\xi_n - a_n| = \lambda_n} \frac{f(\xi)}{\prod_{j=1}^{n} (\xi_j - a_j)^{\alpha_j + 1}} d\xi_1 \cdots d\xi_n$$

$$= \sum_{\alpha} \frac{(D^\alpha f)(a)}{\alpha!} (z - a)^\alpha \quad \text{for } z \in P(a,\mu)$$

Let $\mu, \lambda \to \rho \Rightarrow$ the above holds in $P(a,\rho)$.

Remark: TFAE (The following are equivalent):

(1) f is holomorphic in $P(a,\rho)$;

(2) f can be expanded as a convergent power series at a in $P(a,\rho)$;

(3) $f \in C^0(P(a,\rho))$ and separately holomorphic in each z_k.

(4) f is separately holomorphic in each z_k.

Cauchy's inequalities

Corollary: Let f be holomorphic in $P(a,\rho)$. Assume $|f(z)| \leq M$ in $P(a,\rho)$. Then

$$|D^\alpha f(a)| \leq M \frac{\alpha!}{\rho^\alpha}$$

(Recall $\rho = (\rho_1, \cdots, \rho_n)$, $\alpha = (\alpha_1, \cdots, \alpha_n)$
$\rho^\alpha \triangleq \rho_1^{\alpha_1} \cdots \rho_n^{\alpha_n}$), $\alpha! = \alpha_1! \cdots \alpha_n!$)

Consequently, write $f(z) = \sum_\alpha a_\alpha (z-a)^{\alpha_1}$. Then

$$|a_\alpha| \leq \frac{M}{\rho^\alpha}$$

Pf: E.x. Use 13). The pf is similar to the one variable case.

Thm (Uniqueness Thm) Let Ω be a domain (open and connected) in $\mathbb{C}^n$, $f \in H(\Omega)$. If $f \equiv 0$ on some nonempty open set $E \subseteq \Omega$. Then $f \equiv 0$ on Ω.

Pf: E.x. Hint: Let $G = \{z \in \Omega : D^\alpha f = 0, \forall \alpha\}$.

prove: ① $G \neq \emptyset$

② G is closed in Ω

③ G is open in Ω.

Thm: (open mapping Thm) Let Ω be a domain in $\mathbb{C}^n$. $f \in H(\Omega)$ and f nonconstant. Then f maps any open subset of Ω to an open subset in $\mathbb{C}$.

Pf: Ex. Hint: Restrict f to a complex plane of form:

$$G = \{\lambda \in \mathbb{C} : a + \lambda(b-a) \in \Omega\}$$

where $a, b \in \Omega$ and a, b are close

Then use the open mapping thm in one complex variable

Thm (Maximum principle) Let Ω be a domain in $\mathbb{C}^n$, $f \in H(\Omega)$ and is nonconstant. Then $\sup_{\Omega} |f|$ cannot be achieved at an interior pt of Ω.

Pf: It is a consequence of the open mapping thm.