Math 220A HW 2 Solutions

Chapter 3.1

6. **Find the radius of convergence of each of the following power series:**

   (a) $\sum_{n=0}^{\infty} a^n z^n, a \in \mathbb{C}$**;**

   Solution: To find the radius of convergence R of a power series $\sum_{n=0}^{\infty} a_n z^n$, we generally need to check $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Of course, here $|a_n|^{\frac{1}{n}}$ is just the constant sequence $|a|$, so $R = \frac{1}{|a|}$ (ie, ∞ if $a = 0$). We could also have found this by noting that this series is just a change of variables $z \mapsto az$ of a geometric series, so it converges when $|az| < 1$.

   (b) $\sum_{n=0}^{\infty} a^{n^2} z^n, a \in \mathbb{C}$**;**

   Solution: In this case, $|a_n|^{\frac{1}{n}} = |a|^n$. Therefore, the radius of convergence depends on a, as follows:

   1. if $|a| < 1$, then $\limsup_{n\to\infty} |a|^n = 0$, so $R = \infty$.
   2. if $|a| = 1$, then we are just taking the limsup of $|a|^n = 1$, so $R = 1$.
   3. if $|a| > 1$, then $\limsup_{n\to\infty} |a|^n = \infty$, and $R = 0$.

   (c) $\sum_{n=0}^{\infty} k^n z^n$, k **an integer** $\neq 0$**;**

   Solution: Is this a typo? See part (a).

   (d) $\sum_{n=0}^{\infty} z^{n!}$**.**

   Solution: Here for $n > 1$, $|a_n|^{\frac{1}{n}} = 1$ if $n = k!$ for some non-negative integer k, and 0 otherwise. This means that

$$\begin{aligned}\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} &= \lim_{n\to\infty} \sup_{m\geq n} |a_m|^{\frac{1}{m}} \\ &= \lim_{n\to\infty} \sup\{0,1\},\end{aligned}$$

   and so the radius of convergence is also 1.

Chapter 3.2

1. **Show that $f(z) = |z|^2 = x^2 + y^2$ has derivative only at the origin.**

Solution: Suppose that f is differentiable at some $a \in \mathbb{C}$—ie, that the limit

$$L = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Consider the line ℓ between the origin and a, and take its perpendicular $\ell^\perp$ through a. Let a_t be the point on ℓ at distance t further from the origin than a, and $a_t^\perp$ be either point on $\ell^\perp$ at distance t from a. On the one hand, $|a_t| = |a| + t$ and $h_t := a_t - a = te^{i\theta}$, where $\theta = \arg a$, so

$$\begin{aligned}
\lim_{h_t \to 0} \frac{f(a_t) - f(a)}{h_t} &= \lim_{t \to 0} \frac{(|a|^2 + 2|a|t + t^2 - |a|^2)}{te^{i\theta}} \\
&= \lim_{t \to 0} \frac{2|a| + t}{e^{i\theta}} \\
&= \frac{2|a|}{e^{i\theta}}
\end{aligned}$$

On the other, by the Pythagorean theorem $|a_t^\perp|^2 = |a|^2 + t^2$, while $h_t^\perp := a_t^\perp - a = te^{i(\theta \pm \frac{\pi}{2})}$, meaning

$$\begin{aligned}
\lim_{h_t^\perp \to 0} \frac{f(a_t^\perp) - f(a)}{ht^\perp} &= \lim_{t \to 0} \frac{(|a|^2 + t^2 - |a|^2)}{te^{i(\theta \pm \frac{\pi}{2})}} \\
&= \lim_{t \to 0} \frac{t}{e^{i(\theta \pm \frac{\pi}{2})}} \\
&= 0
\end{aligned}$$

But in both cases were are taking a limit of the difference quotient as z approaches a, so they must both be equal to L. This is a contradiction unless $a = 0$, so it remains merely to check that f actually has a derivative at 0. This is easy, since in that case

$$\begin{aligned}
\lim_{h \to 0} \frac{f(h) - f(0)}{h} &= \lim_{h \to 0} \frac{|h|^2}{h} \\
&= \lim_{h \to 0} \bar{h} \\
&= 0
\end{aligned}$$

4. **Show that $(\cos z)' = -\sin z$ and $(\sin z)' = \cos z$.**

Solution: We can use the formulae

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Differentiating these, using that $(e^{az})' = ae^{az}$ for any constant a, we have

$$\begin{aligned}
(\cos z)' &= \frac{ie^{iz} - ie^{-iz}}{2} \\
&= -\frac{e^{iz} - e^{-iz}}{2i} \\
&= -\sin z
\end{aligned}$$

while

$$\begin{aligned}
(\sin z)' &= \frac{ie^{iz} + ie^{-iz}}{2i} \\
&= \frac{e^{iz} + e^{-iz}}{2} \\
&= \cos z
\end{aligned}$$