1. Consider \( y''' - 2y'' + y = \sin(t) \), rewrite the given scalar equation as a first-order system. Express the system in the matrix form \( \mathbf{x}' = A\mathbf{x} + \mathbf{g} \).

**Answer.** Let \( x_1 = y, x_2 = y' \) and \( x_3 = y'' \), thus

\[
x_1' = y' = x_2, \quad x_2' = y'' = x_3, \quad x_3' = y''' = 2x_3 - x_1 + \sin(t).
\]

In matrix form this is

\[
\begin{pmatrix}
x_1' \\
x_2' \\
x_3'
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} + 
\begin{pmatrix}
0 \\
0 \\
\sin(t)
\end{pmatrix}
\]

2. Solve the initial value problem.

\[
\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}
\]

**Answer.** Let 

\[
A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}
\]

The characteristic equation of \( A \) is:

\[
\det(A - rI) = \det \begin{pmatrix} 1 - r & 2 \\ 3 & 2 - r \end{pmatrix} = (1 - r)(2 - r) - 6 = 0
\]

which is \( r^2 - 3r - 4 = (r + 1)(r - 4) = 0 \). So the eigenvalues of \( A \) are \( r_1 = -1, r_2 = 4 \).

For \( r_1 = -1 \), suppose \( \mathbf{v} \) is an eigenvector, then:

\[
(A - r_1I)\mathbf{v} = 0
\]

which is

\[
\begin{pmatrix}
2 & 2 \\
3 & 3
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} = 
\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

We can take \( \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).
For $r_2 = 4$, suppose $v$ is an eigenvector, then:

$$(A - r_2I)v = 0$$

which is

$$
\begin{pmatrix}
-3 & 2 \\
3 & -2
\end{pmatrix}
\begin{pmatrix} v_1 \\
v_2
\end{pmatrix} =
\begin{pmatrix} 0 \\
0
\end{pmatrix}
$$

We can take $v = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$.

The corresponding eigensolutions are:

$$x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}, \quad x_2(t) = \begin{pmatrix} 1/3 \\ 2 \end{pmatrix} e^{4t}$$

So the general solution is

$$x(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1/3 \\ 2 \end{pmatrix} e^{4t}$$

When $t = 0$, we have $x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ solve for the initial conditions, we will get $c_1 = 1, c_2 = 2$. Therefore the solution to the initial value problem is

$$x(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + 2 \begin{pmatrix} 1/3 \\ 2 \end{pmatrix} e^{4t}.$$  

3. Solve the initial value problem.

$$x' = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} -10 \\ -6 \end{pmatrix}$$

**Answer.** Let

$$A = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix}$$

The characteristic equation of $A$ is:

$$\det(A - rI) = \det \begin{pmatrix} 6 - r & -3 \\ 2 & 1 - r \end{pmatrix} = (6 - r)(1 - r) + 6 = 0$$

which is $r^2 - 7r + 12 = (r - 3)(r - 4) = 0$. So the eigenvalues of $A$ are $r_1 = 3, r_2 = 4$.

For $r_1 = 3$, suppose $v$ is an eigenvector, then:

$$(A - r_1I)v = 0$$
which is
\[
\begin{pmatrix}
3 & -3 \\
2 & -2
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
We can take \( \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

For \( r_2 = 4 \), suppose \( \mathbf{v} \) is an eigenvector, then:
\[
(A - r_2 I)\mathbf{v} = \mathbf{0}
\]
which is
\[
\begin{pmatrix}
2 & -3 \\
2 & -3
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
We choose \( \mathbf{v} = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} \)

The corresponding solutions are:
\[
\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^{4t}
\]

So the general solution is
\[
\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^{4t}
\]

When \( t = 0 \), we have \( \mathbf{x}(0) = \begin{pmatrix} -10 \\ -6 \end{pmatrix} \), solve \( c_1, c_2 \) for the initial conditions, we will get \( c_1 = 2, c_2 = -12 \). Therefore the solution to the initial value problem is
\[
\mathbf{x}(t) = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} - 12 \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^{4t}.
\]

4. Find the general solution of the differential equation:
\[
\mathbf{x}' = \begin{pmatrix}
2 & -1 \\
3 & -2
\end{pmatrix} \mathbf{x} + \begin{pmatrix}
e^{2t} \\
e^{2t}
\end{pmatrix}
\]

**Answer.** Calculating the eigenvalues and eigenvectors for \( \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \) and the two eigensolutions, we will get a fundamental matrix for the corresponding homogeneous linear system:
\[
M(t) = \begin{pmatrix}
e^t & e^{-t} \\
e^t & 3e^{-t}
\end{pmatrix}
\]
Its inverse is

\[
M(t)^{-1} = \frac{1}{2} \begin{pmatrix}
3e^{-t} & e^{t} \\
-e^{t} & e^{t}
\end{pmatrix}
\]

The integral

\[
\int M(t)^{-1} \begin{pmatrix}
e^{2t} \\
1
\end{pmatrix} dt = \frac{1}{2} \int \begin{pmatrix}3e^{t} - e^{-t} \\
e^{-3t} + e^{t}
\end{pmatrix} dt = \frac{1}{2} \begin{pmatrix}3e^{t} + e^{-t} \\
-e^{3t}/3 + e^{t}
\end{pmatrix}
\]

By variation of parameters, we know the general solution for the non-homogeneous equation is:

\[
M(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + M(t) \int M(t)^{-1} \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix} dt
\]

Which is equal to:

\[
\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 4e^{2t}/3 + 1 \\ e^{2t} + 2 \end{pmatrix}
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants.