1. Find a power series solution centered at $x = 0$ to the differential equation $y' + 2y = 0$. Your answer should include a general formula for the coefficients. Then explain why your solution is the same as $y = a_0 e^{-2x}$.

**Answer.** Assume $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Plug into the equation

$$y' + 2y = \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$ 

Shifting the summation index, we shall get

$$\sum_{k=0}^{\infty} (k + 1) a_{k+1} x^k + 2 \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\Rightarrow \sum_{k=0}^{\infty} (k + 1) a_{k+1} x^k + 2 \sum_{k=0}^{\infty} a_k x^k = 0,$$

thus the recurrence relation is

$$(k + 1) a_{k+1} + 2a_k = 0, \quad k = 0, 1, 2, \ldots$$

and $a_{k+1} = \frac{-2a_k}{k+1}$. The general formula for $a_k$ is

$$a_k = \frac{(-2)^k}{k!} a_0, \quad k = 0, 1, 2, \ldots$$

Therefore the power series solution for the differential equation is

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} x^k.$$ 

Note the Taylor series of $e^t$ is $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$. Replacing $t$ by $-2x$, we get $e^{-2x} = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} x^k$. Hence the solution $y$ is the same as $y = a_0 e^{-2x}$.

2. Solve the initial value problem using a power series centered at $x = 0$. Write out the first four nonzero terms of the infinite series:

$$y'' - xy' - y = 0, \quad y(0) = 2, \; y'(0) = -1.$$
Answer. Assume \( y = \sum_{n=0}^{\infty} a_n x^n \), then \( y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \) and \( y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \). Plug into the equation

\[
y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0,
\]

\[
\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0
\]

Shifting the summation index, we shall get

\[
\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0,
\]

thus the recurrence relation is \( 2a_2 - a_0 = 0 \) and

\[
(k+2)(k+1) a_{k+2} - k a_k - a_k = 0, \quad k = 1, 2, \ldots,
\]

thus \( (k+2) a_{k+2} = a_k \). The first few terms are

\[
a_3 = \frac{1}{3} a_1, \quad a_4 = \frac{1}{8} a_0, \quad a_5 = \frac{1}{15} a_1, \quad \ldots
\]

Therefore

\[
y = \sum_{n=0}^{\infty} a_n x^n = a_0 (1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \cdots) + a_1 (x + \frac{1}{3} x^3 + \frac{1}{15} x^5 + \cdots).
\]

Since \( y(0) = a_0 \) and \( y'(0) = a_1 \), we get \( a_0 = 2 \) and \( a_1 = -1 \). The first four nonzero terms of the power series solution is

\[
y = 2 - x + x^2 - \frac{1}{3} x^3 + \cdots.
\]

3. Solve the initial value problem using a power series centered at \( x = 0 \). Write out the first four nonzero terms of the infinite series:

\[
(1 - x)y'' + y = 0, \quad y(0) = 3, \quad y'(0) = 0.
\]

Answer. Assume \( y = \sum_{n=0}^{\infty} a_n x^n \), then \( y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \) and \( y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \). Plug into the equation

\[
(1 - x)y'' + y = (1 - x) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0,
\]
\[
\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0,
\]

\[
\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.
\]

Shifting the summation index, we shall get

\[
\sum_{k=0}^{\infty} (k + 2)(k + 1)a_{k+2} x^k - \sum_{k=1}^{\infty} (k + 1)k a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^k = 0,
\]

thus the recurrence relation is \(2a_2 + a_0 = 0\) (thus \(a_2 = -\frac{1}{2}a_0\)) and

\[(k + 2)(k + 1)a_{k+2} - (k + 1)k a_{k+1} + a_k = 0, \quad k = 1, 2, \ldots .\]

The first few terms are

\[a_3 = \frac{1}{3} a_2 - \frac{a_1}{6} \Rightarrow a_3 = -\frac{1}{6}a_0 - \frac{a_1}{6},\]

\[a_4 = \frac{1}{2}a_3 - \frac{1}{12}a_2 \Rightarrow a_4 = -\frac{1}{24}a_0 - \frac{1}{12}a_1,\]

Therefore

\[y = \sum_{n=0}^{\infty} a_n x^n \]

\[= a_0 + a_1 x - \frac{a_0}{2} x^2 - \left(\frac{1}{6}a_0 + \frac{1}{6}a_1\right)x^3 - \left(\frac{1}{24}a_0 + \frac{1}{12}a_1\right)x^4 + \cdots .\]

Since \(y(0) = a_0\) and \(y'(0) = a_1\), we get \(a_0 = 3\) and \(a_1 = 0\). The first four nonzero terms of the power series solution is

\[y = 3 - \frac{3}{2} x^2 - \frac{1}{2} x^3 - \frac{1}{8} x^4 + \cdots .\]