Lecture 6

Plan of Lecture 6

4.2

§ 4.2 Homogeneous Linear Equation (2nd order)

Def.: By a linear 2nd order constant coefficient D.E., we mean

\[ ay'' + by' + cy = f(x) \] (1)

Here a, b, c are constants and a ≠ 0.

If \( f = 0 \), then (1) becomes

\[ ay'' + by' + cy = 0. \]

we say it is homogeneous.
If $f \neq 0$

we say it is nonhomogeneous

Remark: There are many examples of linear 2nd order constant coefficient D.IEs in real life.

E.g. A mass on a spring

\[ m = \text{mass} \]
\[ k = \text{spring constant} \]

\[ x_0, \text{ rest position of the spring} \]
\[ x, \text{ the position of the mass} \]

At time $t$

\[ x(t) = \text{position of the mass at } t. \]
Newton's second law:

\[ F = ma \]

- force
- mass
- acceleration

Hooke's law:

force exerted by the spring = \( k \cdot x \)

\[ \Rightarrow F = -kx \]

Note: \( a = \frac{d^2x}{dt^2} = x'' \)

Remark: when \( x > 0 \), \( F \) is negative pointing.

\[ F = ma \Rightarrow -kx = mx'' \]

or \( mx'' + kx = 0 \)
Q: How to solve
\[ ay'' + by' + cy = f(x) \] ?

Today we will consider the easier case where \( f = 0 \), i.e. the homogeneous case:
\[ ay'' + by' + cy = 0 \] (1)

Let's first discuss some properties of (1).

**Properties**

1. If \( y_1(x) \) is a solution of (1), \( C_1 \) is constant, then \( C_1y_1(x) \) is also a solution of (1).

2. If \( y_1(x) \) and \( y_2(x) \) are both solutions of (1), then \( y_1(x) + y_2(x) \) is also a solution of (1).
(3) If $y_1(x)$ and $y_2(x)$ are solutions of (*), $c_1, c_2$ are constants, then

$$c_1 y_1(x) + c_2 y_2(x)$$

is a solution of (*).

**Proof.**

1. Since $y_1$ is a solution $\Rightarrow$

$$a y_1'' + b y_1' + c y_1 = 0$$

Let's check $c_1 y_1$:

$$\text{LHS} = a (c_1 y_1)'' + b (c_1 y_1)' + c (c_1 y_1)$$

$$= c_1 (a y_1'' + b y_1' + c y_1) = 0$$

(2) Since $y_1, y_2$ are solutions, $\Rightarrow$

$$\begin{cases}
ay_1'' + by_1' + cy_1 = 0 \\
ay_2'' + by_2' + cy_2 = 0
\end{cases}$$

$$y_1 + y_2 \Rightarrow a(y_1 + y_2)'' + b(y_1 + y_2)' + c(y_1 + y_2) = 0$$
$\Rightarrow y_1 + y_2$ is a soln!

3. Exercise!

\[ \begin{align*}
  &y_1 = e^x \\
  &y_1' = e^x \\
  &y_1'' = e^x
\end{align*} \]

\[ \begin{align*}
  \text{Ex.} & : \quad y_1(x) = e^x \text{ is a soln of } y'' = y. \\
  \text{Then } & \pm e^x \text{ is also a soln} \\
  10e^x & \text{ is also a soln.} \\
  \text{every } & \pm Ce^x \text{ is also a soln for } C \in \mathbb{R}
\end{align*} \]

Let's come back to

Q: How to solve $ay'' + by' + cy = 0$?

A: Let's first try simply ones:

Find out one soln of:

1. $y'' - y = 0$ \quad (a=1, \quad b=0, \quad c=-1)

\[ \implies y'' = y \]

one soln: $y_1 = e^x$
\[ y'' - 4y = 0 \iff y'' = 4y \]

One solution: \( y_1 = e^{2x} \)

\[ y'' - 9y = 0 \iff y'' = 9y \]

One solution: \( y_1 = e^{3x} \)

4. In general,
\[ y'' = A^2 y, \ A \in \mathbb{R} \]

has a solution: \( y_1 = e^{Ax} \)

Can we use the same idea to solve
\[ ay'' + by' + cy = 0 \quad (x) \quad ? \]
\[
(a \neq 0) \]

Yes! The idea is: Try \( e^{\lambda x} \).
\[ \lambda \text{ is a constant to be determined!} \]
That is, suppose \( y = e^{\lambda x} \) solves
\[ ay'' + by' + cy = 0 \]
Then
\[ a(\lambda^2 e^{\lambda x}) + b(\lambda e^{\lambda x}) + c(e^{\lambda x}) = 0 \]
\[ \Rightarrow e^{\lambda x} (a\lambda^2 + b\lambda + c) = 0 \]

Q: How to make the above hold?
A: \( e^{\lambda x} \) cannot be zero, we thus need
\[ a\lambda^2 + b\lambda + c = 0 \]

*Def*:
\[ a\lambda^2 + b\lambda + c = 0 \]

is called the characteristic eqn of (★)

E.g
\[ y'' - 5y' + 6y = 0 \]  \hspace{1cm} (2)

Assume \( y = e^{\lambda x} \) is a soln we need
\[ \lambda^2 - 5\lambda + 6 = 0 \]
\[ (\lambda - 2)(\lambda - 3) = 0 \]
\[ \Rightarrow \lambda_1 = 2, \lambda_2 = 3. \]

Thus we obtain two solns, \( e^{2x}, e^{3x} \)
(you can verify they are both solns!)
Check: \( e^{2x} \)

\[
\left( e^{2x} \right)^{\prime\prime} - 5(e^{2x})' + 6 e^{2x} = 4 e^{2x} - 10 e^{2x} + 6 e^{2x} = 0
\]

\( e^{3x} \) Exercise.

By properties: for any \( c_1, c_2 \in \mathbb{R} \),

\[ c_1 e^{2x} + c_2 e^{3x} \] is a soln!

Idea: In general, to solve

\[ ay'' + by' + cy = 0, \quad a \neq 0 \quad (1) \]

we need to solve \( a\lambda^2 + b\lambda + c = 0 \).

If \( \lambda \) satisfies \( a\lambda^2 + b\lambda + c = 0 \), then

\( e^{\lambda x} \) is a soln of (1).

Recall quadratic formula:

\[ a\lambda^2 + b\lambda + c = 0, \quad a \neq 0 \]

\[ \Rightarrow \lambda = \frac{-b \pm \sqrt{\Delta}}{2a} \quad \Delta = b^2 - 4ac \]

Three cases:

(I) \( \Delta > 0 \), two distinct real roots, \( \lambda_1 \neq \lambda_2 \)

(II) \( \Delta = 0 \), one repeated real root, \( \lambda_1 = \lambda_2 \)

(III) \( \Delta < 0 \), no real roots.
ay'' + by' + cy = 0 \hspace{1cm} (\star)

Now two questions.

Q1: How to find all solutions to (\star)?

Q2: Are the treatments for cases (I), (II), (III) the same?

For example, in (III), there are no roots!

What should we do?

A to Q1:

Def.: Let \( y_1(x) \) and \( y_2(x) \) be two functions on an interval \( I \).

- We say \( y_1 \) and \( y_2 \) are linearly dependent \((\text{"L.D."})\) if one of them is a constant multiple of the other \((y_1 = ky_2 \text{ or } y_2 = ky_1)\).
- We say \( y_1 \) and \( y_2 \) are linearly independent \((\text{"L.I."})\) if neither of them is a constant multiple to the other.
How to check whether $y_1, y_2$ are L.I or L.D?

**Thm.** Define the Wronskian of $y_1, y_2$ to be

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

Then $y_1, y_2$ are L.D. on an interval $I$ if $W(y_1, y_2) = 0$ everywhere on $I$.

This means:

1. If $W(y_1, y_2)$ is everywhere 0, $\Rightarrow$ $y_1, y_2$ are L.D.
2. If $W(y_1, y_2)$ is not everywhere 0 $\Rightarrow$ $y_1, y_2$ are L.I

**Example:** Let $y_1 = x + 1$, $y_2 = e^x$

1. Verify $W(y_1, y_2) = xe^x$ **Exerice**

2. Since $W(y_1, y_2) = 0$ only at one point $x = 0$
\[ \Rightarrow y_1, y_2 \text{ is L.I on } \mathbb{R}. \]

Example 2: Let \( y_1 = x, \ y_2 = 2x \)

\[ \Rightarrow y_1, y_2 \text{ are L.D. (} y_2 = ky_1, \text{ with } k = 2) \]

Can we check by Wronskian?

Yes, \( W(x, 2x) = y_1y_2' - y_2y_1' \)

\[ = x \cdot 2 - 2x \cdot 1 = 0 \text{ everywhere} \]

Remark:

1. If \( \lambda_1 \neq \lambda_2 \), then \( e^{\lambda_1 x}, e^{\lambda_2 x} \) are L.I
2. If \( \lambda_1 = \lambda_2 \), then \( e^{\lambda_1 x}, e^{\lambda_2 x} \) are L.D.

Why? Use Wronskian

\[ W(y_1, y_2) = e^{\lambda_1 x} (e^{\lambda_2 x})' - (e^{\lambda_2 x})' (e^{\lambda_1 x})' \]

\[ = (\lambda_2 - \lambda_1) e^{\lambda_1 x} e^{\lambda_2 x} \]

\[ = 0 \text{ if } \lambda_1 = \lambda_2 \]

\[ \neq 0 \text{ if } \lambda_1 \neq \lambda_2 \]

**Thm:** Given

\[ ay'' + by' + cy = 0 \quad (1) \]

If \( y_1, y_2 \) are two L.I solns of (1)

then \( c_1 y_1 + c_2 y_2 \) gives all solns of (1)

Called the general solns of (1)

where \( c_1, c_2 \in \mathbb{R} \)
This means, every solution of (1) can be obtained from the form \(c_1 y_1 + c_2 y_2\).

**Remark:** By the Thm, to find the general solutions of (1), we just need to find two L.I. solutions of (1).

Now consider the case \(\Delta = b^2 - 4ac > 0\).

Then \(a \lambda^2 + b\lambda + c = 0\)

has two distinct solutions \(\lambda_1, \lambda_2\) \((\lambda_1 \neq \lambda_2)\).

\(y_1 = e^{\lambda_1 x}\) and \(y_2 = e^{\lambda_2 x}\) are both solutions of

\(a y'' + b y' + c y = 0\) \((1)\)

And \(y_1\) and \(y_2\) are "L.I."!!

By the Thm,

\(c_1 y_1 + c_2 y_2 = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\)

gives all the solutions of \((1)\)
Consider \( y'' - 5y' + 6y = 0 \) \hspace{1cm} (2)

**Step 1:** Solve the characteristic eqn:

\[ \lambda^2 - 5\lambda + 6 = 0 \]

\[ (\lambda - 2)(\lambda - 3) = 0 \]

\[ \Rightarrow \] Two distinct roots: \( \lambda_1 = 2, \lambda_2 = 3 \).

\[ \Delta = b^2 - 4ac \]

\[ = 25 - 4 \times 1 \times 6 \]

\[ = 1 > 0 \]

**Step 2:** If there are two distinct solns \( \lambda_1, \lambda_2 \) in step 1, then there are two L.I. solns to (3): \( e^{\lambda_1 x}, e^{\lambda_2 x} \)

\[ y_1(x) = e^{2x}, \hspace{0.5cm} y_2(x) = e^{3x} \]

The general soln:

\[ y = c_1 y_1 + c_2 y_2 \]

\[ = c_1 e^{2x} + c_2 e^{3x} \]
Now what if \( \Delta = 0 \)?

E.g. \( y'' + 4y' + 4y = 0 \)

Characteristic eqn: \( \lambda^2 + 4\lambda + 4 = 0 \)

\[ \Rightarrow \Delta = 4^2 - 4 \times 4 = 0. \]

It has only one repeated root

\[ \lambda = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-4}{2} = -2. \]

We can get in this way only one soln. \( e^{-2x} \)

How to get another soln?

Thm: If the characteristic eqn

\[ a\lambda^2 + b\lambda + c = 0, \]

has only one repeated root \( \lambda_0 \).

Then \( e^{\lambda_0 x} \), \( xe^{\lambda_0 x} \) are two L.I.
sols, and the general solns are
\[ C_1 e^{\lambda_1 x} + C_2 xe^{\lambda_2 x} \text{ where } C_1, C_2 \in \mathbb{R} \]
\[ \text{check: } W(e^{\lambda_1 x}, xe^{\lambda_2 x}) \neq 0 \]

Recall
\[ y'' + 4y' + 4y = 0 \quad (3) \]
The characteristic eqn
\[ \lambda^2 + 4\lambda + 4 = 0 \]
has a repeated root \( \lambda = -2 \).

Then (3) has two L.I. solns.
\[ y_1 = e^{-2x}, \quad y_2 = xe^{-2x} \]

Exercise: check \( y_2 \) is a soln of (3)!
\[ y_2' = e^{-2x} - 2xe^{-2x} \]
\[ y_2'' = -2e^{-2x} - 2e^{-2x} + 4xe^{-2x} \]
\[ \Rightarrow \quad y'' + 4y' + 4y = 0 \]

The general soln of (3) is

\[ y = c_1 e^{2x} + c_2 xe^{2x}. \]