Plan of Lecture 5:

- Very quick review of §2.4
- §2.5

Review of §2.4

- "M(x,y)dx + N(x,y)dy = 0" is exact
  \[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

- If
  \[ dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy \]
  
  then "Mdx + Ndy = 0" has an implicit solution
  \[ F(x,y) = C \]
When \( Mdx + Ndy = 0 \) is exact, how to find \( F \) s.t. \( dF = Mdx + Ndy \)?

We need to find \( F \) s.t.

\[
\begin{align*}
\frac{\partial F}{\partial x} &= M \\
\frac{\partial F}{\partial y} &= N
\end{align*}
\]

Step 1: Integrate (1) w.r.t. \( x \) by regarding \( y \) as a constant

Step 2: Substitute what we get in step 1 into (2)

\[ \Delta 2.5 \text{ Integrating factor} \]

Q: What if \( Mdx + Ndy = 0 \) is NOT exact? Can we still solve it?

A: For some “nice” non-exact equation, yes!
Indeed, for some non-exact equation

$$M \, dx + N \, dy = 0, \quad (1)$$

we can "manually" make it exact by multiplying some function $\mu$. That is,

$$\left( \frac{\mu(x,y) \, M(x,y) \, dx}{M_1} + \frac{\mu(x,y) \, N(x,y) \, dy}{N_1} \right)$$

is exact.

In this case, $\mu$ is called an integrating factor of (1).

Q: What is the relation between "integrating factor" here and "integrating factor" in §2.3?

A: We will discuss at the end. In short, they are different but highly similar in spirit.
First let's consider the following Q:

Q: How to find \( \mu(x,y) \) in (2)?

A: Let \( M_1 = \mu M \)

\[
N_1 = \mu N
\]

(2) is exact \( \iff \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \)

This means

\[
\frac{\partial}{\partial y} [\mu M] = \frac{\partial}{\partial x} [\mu N]
\]

\[\Rightarrow \partial_y \mu M + \mu \partial_y M = \partial_x \mu N + \mu \partial_x N\]

\[\Rightarrow M \partial_y \mu - N \partial_x \mu = (\partial_x N - \partial_y M) \mu \quad (3)\]

In general, it is very difficult to solve (3).

But in some exceptional cases, we can solve it.

(II) Suppose we can make \( \partial_y \mu = 0 \). Then (3) \( \Rightarrow \)

\[-N \partial_x \mu = (\partial_x N - \partial_y M) \mu\]
\[ \Delta x \mu = \frac{1}{N} (\Delta y M - \Delta x N) \mu \quad (4) \]

Suppose \( \frac{1}{N} (\Delta y M - \Delta x N) \) only depends on \( x \).

Call it \( f(x) \). Then (4) implies:

\[ \frac{1}{\mu} \frac{d\mu}{dx} = \frac{1}{\mu} \Delta x \mu = f(x) \]

\[ \Rightarrow \quad \frac{1}{\mu} d\mu = f(x) \, dx \]

\[ \Rightarrow \quad \int \frac{1}{\mu} d\mu = \int f(x) \, dx \]

\[ \Rightarrow \quad \ln |\mu| = \int f(x) \, dx \]

\[ \Rightarrow \quad |\mu| = e^{\int f(x) \, dx} \Rightarrow \mu = \pm e^{\int f(x) \, dx} \]

Both of \( \pm \) will work. We always choose the \( + \) one. \( \Rightarrow \mu = e^{\int f(x) \, dx} \) depends only on \( x \).

Summarize:

If \( \frac{1}{N} (\Delta y M - \Delta x N) \) depends on \( x \), then \( \mu = e^{\int f(x) \, dx} \) works! Only

(II) Suppose we can make \( \Delta x \mu = 0 \). Then

(13) becomes
\[ W \partial_y \mu = (\partial_x N - \partial_y W) \mu \]

\[ \Rightarrow \partial_y \mu = \frac{1}{\mu} (\partial_x N - \partial_y W) \mu \quad (5) \]

Suppose \( W(\partial_x N - \partial_y W) \) depends only on \( y \),

Call it \( g(y) \). \( (5) \Rightarrow \)

\[ \frac{1}{\mu} \frac{d\mu}{dy} = \frac{1}{\mu} \partial_y \mu = g(y) \]

\[ \Rightarrow \int \frac{1}{\mu} d\mu = \int g(y) dy \]

\[ \Rightarrow \ln |\mu| = \int g(y) dy \Rightarrow \]

\[ \mu = \pm e^{\int g(y) dy} \]

we normally take the "+" one \( \Rightarrow \)

\[ \mu = e^{\int g(y) dy} = e^{\int \frac{1}{\nu} (\partial_x N - \partial_y W) dy} \]

Note this \( \mu \) is a function depending only on \( y \) ! \( \Rightarrow \frac{\partial \mu}{\partial x} = 0. \]

Summarize: If \( \frac{1}{\nu} (\partial_x N - \partial_y W) \) depends only on \( y \), then \( \mu = e^{\int g(y) dy} \) works!
We summarize the above to the following theorem:

Thm. Given \( Mdx + Ndy = 0 \) \((*)\)

(I) If \( \frac{1}{N} (\partial yM - \partial xN) \) depends only on \( x \)

(that is, does not depend on \( y \)), then

\[
\mu = \mu(x) = e^{\int \frac{1}{N} (\partial yM - \partial xN) \, dx}
\]

is an integrating factor of \((*)\)

(means "\( \mu Mdx + \mu Ndy = 0 \) is exact")

(II) If \( \frac{1}{M} (\partial xN - \partial yM) \) only depends on \( y \)

(that is, does not depend on \( x \)), then

\[
\mu = \mu(y) = e^{\int \frac{1}{M} (\partial xN - \partial yM) \, dy}
\]

is an integrating factor of \((*)\)

(means "\( \mu Mdx + \mu Ndy = 0 \) is exact")

Remark: only use the above theorem when \((*)\) is not exact.
E.g. Solve

\[(x^2 + y)dx + (x^2y - x)dy = 0. \quad (6)\]

\[M = \frac{x^2 + y}{x^2y - x} \quad N = \frac{x^2y - x}{x^2 + y}\]

A:

\[\text{Note } \int M = 2x^2 + y \]
\[\int N = x^2y - x\]

Step 1: Check the exactness of (6).

\[\frac{\partial M}{\partial y} = 1\]
\[\frac{\partial N}{\partial x} = 2xy - 1\]

\[\Rightarrow \frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} \neq \text{not - exact!}\]

Step 2. Can we use the theorem? Need to check!

(I) Compute

\[\frac{1}{N} (\frac{\partial y M - \partial x N}{})\]

\[= \frac{1 - (2xy - 1)}{x^2y - x} = \frac{-2(xy - 1)}{x(xy - 1)} = \frac{-2}{x}\]
It only depends on $x$!

Hence, yes, we can use Theorem (I).

The integrating factor
\[ \mu(x) = e^{\int -\frac{3}{x} \, dx} \]
\[ = e^{-3 \ln|x| + C} \]
\[ = e^{\frac{1}{x^3}} = \frac{1}{x^3} = x^{-3} \]

Choose $C = 0$

\[ = \frac{1}{x^3} = \frac{1}{x^2} = x^{-2} \]

Step 3. Multiply both sides by $\mu(x)$

\[ x^{-2} (2x^2 + y) \, dx + x^{-2} (x^2y - x) \, dy = 0 \]

\[ \Rightarrow (2 + \frac{y}{x^2}) \, dx + (y - \frac{x}{x}) \, dy = 0 \]

This new D.E is exact!

\[ \text{use last lecture} \]

Step 4. We next find $F$ s.t

\[ \int \frac{\partial F}{\partial x} = 2 + \frac{y}{x^2} \quad \text{①} \]

\[ \int \frac{\partial F}{\partial y} = y - \frac{1}{x} \quad \text{②} \]
Integrating (1) and regarding $y$ as a constant

$$\Rightarrow \quad F = \int \frac{df}{dx} \, dx = \int \left( 2 + \frac{y}{x^2} \right) \, dx$$

$$= 2x - \frac{y}{x} + g(y) \quad \text{constant term}$$

Substitute into (2) ⇒

$$-\frac{1}{x} + g'(y) = \frac{dF}{dy} = y - x^{-1}$$

$$\Rightarrow \quad g'(y) = y \Rightarrow g(y) = \frac{1}{2} y^2$$

Hence $F = 2x - \frac{y}{x} + \frac{1}{2} y^2$

The solution to the D.E. is $(F = C)$

$$2x - \frac{y}{x} + \frac{1}{2} y^2 = C.$$
Q: What the relation between "integrating factor" here and "integrating factor" in § 2.3?

Recall "integrating factor" in § 2.3:

\[ \frac{dy}{dx} + p(x)y = Q(x) \]

The integrating factor \( \mu(x) = e^{\int p(x)dx} \)

Sometimes, you can both of the "integrating factor" methods to solve a D.E.

E.g. Solve

\[ \frac{dy}{dx} + \frac{3}{x} y = 2, \quad x > 0. \quad (7) \]

Way 1: (Use § 2.3)

Note \( \int p(x) = \frac{3}{x} \)

\[ \int Q(x) = 2 \]

\[ \Rightarrow \mu = e^{\int p(x)dx} = e^{\int \frac{3}{x}dx} = x^3 \]

\[ e^{3\ln|x|} \]
Multiply $\mu$ to both sides $\Rightarrow$

$$x^3 \frac{dy}{dx} + 3x^2 y = 2x^3$$

$$\Rightarrow \frac{d}{dx}(x^3 y) = 2x^3$$

$$\Rightarrow x^3 y = \int 2x^3 \, dx = \frac{1}{2}x^4 + C$$

$$\Rightarrow y = \frac{1}{2}x + \frac{C}{x^3}$$

Way 2 (use today's lecture)

Reduce (7) to the form \[ M \, dx + N \, dy = 0 \]

(7) $\Rightarrow$ dy = (2 - $\frac{3y}{x}$) dx

$\Rightarrow \left( \frac{3y}{x} - 2 \right) \, dx + dy = 0 \quad (8)$

$\Rightarrow \left\{ \begin{array}{l}
M = \frac{3y}{x} - 2 \\
N = 1
\end{array} \right.$

Step 1: check exactness

$$\left\{ \begin{array}{l}
\frac{\partial M}{\partial y} = \frac{3}{x} \quad \Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \\
\frac{\partial N}{\partial x} = 0
\end{array} \right.$$
Step 2. Try to use the theorem.

Check which case it is ((I) or (II))

Check (I). Compute

\[
\frac{1}{N} (\partial y M - \partial x N)
\]

\[
= \frac{1}{1} \left( \frac{3}{x} - 0 \right) = \frac{3}{x}
\]

It only depends on \( x \)!

Use Case (I).

\[
\mu = e^{\int \frac{3}{x} \, dx} = e^{3 \ln|x|} = 1x^3 = x^3
\]

Multiply (18) by \( \mu \) \( \Rightarrow \)

\[
(3x^2y - 2x^3) \, dx + x^3 \, dy = 0
\]
This new P.E is exact!

We will find \( F \) s.t.

\[
\begin{align*}
\frac{\partial F}{\partial x} &= 3x^2y - 2x^3 \quad (1) \\
\frac{\partial F}{\partial y} &= x^3 \quad (2)
\end{align*}
\]

Integrating (1) \( \Rightarrow \)

\[
F = \int (3x^2y - 2x^3) \, dx
\]

\[
= x^3y - \frac{1}{2}x^4 + g(y)
\]

Substitute \( F \) into (2)

\[
x^3 + g' = \frac{\partial F}{\partial y} = x^3
\]

\( \Rightarrow \)

\[
g' = 0 \quad \Rightarrow \quad g = c
\]

Choose \( c = 0 \) \( \Rightarrow \quad g = 0
\]

\( \Rightarrow \quad F = x^3y - \frac{1}{2}x^4 \)
The soln to the D.E is

\[ x^3 y - \frac{1}{2} x^4 = C \]