Recall: Rough speaking, a D.E of order $n$

is

$$F(x, y, \ldots, \frac{d^ny}{dx^n}) = 0 \quad (*)$$

Two definitions:

- explicit solution
- implicit solution.

Def.: By an explicit solution to $(*)$ we mean a function $y = \phi(x)$ that satisfies $(*)$.

E.g.

1. Given D.E

$$\frac{d^2y}{dx^2} + y = 0 \quad (1)$$

Verify for all $A, B \in \mathbb{R}$.

$y = A \cos x + B \sin x$ is an explicit solution to $(1)$. 
② Find values of A, B such that
\[ y = A \cos x + B \sin x \]
Solves I.V.P:
\[
\begin{cases}
\frac{d^2y}{dx^2} + y = 0 & \text{D.E} \\
y(0) = 1, \quad y'(0) = 2 & \text{initial condition}
\end{cases}
\]

\[
A = 0 \quad \text{(plug in } y = \phi(x) \text{ and verify the D.E holds)}
\]

\[
\begin{align*}
\text{Since } y &= A \cos x + B \sin x \\
\Rightarrow \quad \frac{dy}{dx} &= y' = -A \sin x + B \cos x \\
\frac{d^2y}{dx^2} &= (y')' = -A \cos x - B \sin x \\
\text{Hence } \frac{d^2y}{dx^2} + y &= (A \cos x - B \sin x) + (A \cos x + B \sin x) \\
&= 0
\end{align*}
\]
This verifies the D.E holds. and thus $y = A\cos x + B\sin x$ is an explicit soln.

(2) \[(\text{Initial condition: } y(0) = 1, \ y'(0) = 2)\]

Let $x = 0$,

$y(0) = A\cos 0 + B\sin 0$

$= A$

$y(0) = 1 \Rightarrow A = 1$

Recall $y' = -A\sin x + B\cos x$

Let $x = 0$ \Rightarrow

$y'(0) = B$

$y'(0) = 2 \Rightarrow B = 2$

Hence

$y = \cos x + 2\sin x$
Def. A relation/equation \( G(x,y) = 0 \) is called an implicit solution if it gives one or more solutions to the D.E.

Ex. \( x^2 + y^2 = 1 \) gives an implicit solution for the D.E.

\[
\frac{dy}{dx} = -\frac{x}{y}.
\]

Remark: \( x^2 + y^2 = 1 \) \( \Rightarrow \) \( y^2 = 1 - x^2 \) \( \Rightarrow \)

\[ y = \pm \sqrt{1-x^2} \]

(1) \( y = \sqrt{1-x^2} \)

(2) \( y = -\sqrt{1-x^2} \)

Note: In general, it might be hard to solve \( y \) out of the eqn \( G(x,y) = 0 \).

We, however, have a more general way to verify the implicit solution by using implicit differentiation.
\[ y = y(x) \]

\textbf{Step 1:} Regard }y\text{ as a function of }x\text{ and differentiate the eqn with respect to }x.

\[ \frac{d}{dx}(y(x))^2 = 2y \cdot y' \]

\[ \Rightarrow \quad 2x + 2y \frac{dy}{dx} = 0 \]

\[ \Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y} \]
Overview. Chapter 2 discusses 3 ways to solve 1st order D.E.

Sec 2.2 separable D.Es.

A D.E is called separable if it can be rewritten in the form: \( \frac{dy}{dx} = f(x) \cdot g(y) \). Here \( f, g \) are allowed to be constant (that is, RHS of the eqn is the product of two parts, where one part depends only on \( x \), the other part depends only on \( y \)).

E.g. Are the following D.E separable.

1. \( \frac{dy}{dx} - y^2 = 0 \)  

   \[ \Rightarrow \frac{dy}{dx} = y^2 = 1 \cdot y^2 \]
2. \[ \frac{dy}{dx} - y^2 = x \quad \text{Not separable} \]
\[
\frac{dy}{dx} = x + y^2
\]
\[
\frac{?}{?} = f(x) \cdot g(y) \quad \text{Not possible}
\]

3. \[ \frac{dy}{dx} - xy - x - y = 1 \quad \checkmark \text{Separable} \]
\[
\Rightarrow \quad \frac{dy}{dx} = xy + x + y + 1
\]
\[
= (1+x) + y(1+x)
\]
\[
= (1+x)(1+y)
\]
\[
\frac{f(x)}{g(y)}
\]
Ideas to solve separable D.E.

\[ \frac{dy}{dx} = f(x)g(y). \quad (2) \]

Step 1: Check whether \( g(y) = 0 \) gives a soln.

to (2) (If \( g(y) \) cannot be 0, then you don’t need to do step 1)

Step 2: Suppose \( g(y) \neq 0 \).

\[(2) \Rightarrow \frac{dy}{g(y)} = f(x)dx \]

Then integrate \( \Rightarrow \)

\[ \int \frac{dy}{g(y)} = \int f(x)dx \]

Calculus \( \Rightarrow \)

\[ G(y) = F(x) + C \]

Remark 1. The above is an implicit solution (not in the form \( y = \phi(x) \))

2. Technically \( G(y) + C_1 = F(x) + C_2 \)
\[ G(y) = \text{Fix} + \left( \frac{C_2 - C_1}{C} \right) \]

**Eg 1:** Solve \( \frac{dy}{dx} = \frac{y-1}{x+3} \) \hspace{1cm} (3)

**Idea:** \[ \frac{dy}{dx} = \frac{y-1}{x+3} \]
\[ = \frac{1}{x+3} \left( \frac{y-1}{g(y)} \right) \]

**Step 1:** Check whether \( y-1 = 0 \) gives a soln.

\[ y-1 = 0 \Rightarrow y = 1 \text{ Then } \frac{dy}{dx} = 0; \]

LHS of (3) = 0,

RHS of (3) = \( \frac{y-1}{x+3} = 0 \Rightarrow (3) \text{ holds} \)

Hence \( y = 1 \) is a soln to (3).

**Step 2:** Assume \( y-1 \neq 0 \). Move terms about \( y \) to LHS, move terms about \( x \) to RHS

\[ \frac{dy}{y-1} = \frac{1}{x+3} \; dx \]
Integrate \( \Rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x+3} \)

\[ \Rightarrow \]

\[ \ln |y-1| = \ln |x+3| + C \]

(Implicit so \( \ln \)

In particular, \( a=1 \)

\[ \int \frac{1}{at+b} \, dt = \frac{1}{a} \ln |at+b| + C \]

To summarize, we have the following solutions:

1. \( y = 1 \)

2. \( \ln |y-1| = \ln |x+3| + C \)

verify by \( u-\text{sub} \)

"\( u = at+b \)"
E.g 2: solve the following I.V.P

\[ \int \frac{dy}{dx} = \frac{y-1}{x+3} \]  \hspace{1cm} (3)

\[ y(-1) = 0 \]  \hspace{1cm} (4) \hspace{1cm} \text{Initial condition}

Write the soln as an explicit soln \( y = f(x) \).

A: Recall by the above E.g 1, (3) has two kinds of solns:

1. \( y = 1 \) \( \Rightarrow \) \( y = 1 \) everywhere

2. \( \ln|y-1| = (x\ln(x+3) + C \) .

For 1, it can never satisfy \( y(-1) = 0 \)

For 2, since \( y(-1) = 0 \), that is

when \( x = -1 \), \( y = 0 \), we plug in

\( (x, y) = (-1, 0) \) to 2 \( \Rightarrow \)
\[
\ln l - \ln 1 = \ln l - 1 + 3 + C
\]
\[
\Rightarrow \ln l = \ln 2 + C
\]
\[
\Rightarrow C = -\ln 2
\]

Hence the solution is
\[
\ln |y - 1| = \ln |x + 3| - \ln 2
\]

we raise both side to exp.
\[
\Rightarrow e^{\ln |y - 1|} = e^{\ln |x + 3| - \ln 2}
\]

\[e^{\ln a} = a \quad (a > 0)\]
\[e^{A - B} = \frac{e^A}{e^B}\]
\[
\Rightarrow 1_{y - 1} = \frac{e^{\ln |x + 3|}}{e^{\ln 2}}
\]
\[
= \frac{1}{2} |x + 3|
\]
\[
\Rightarrow 1_{y - 1} = \frac{1}{2} |x + 3|
\]
\[ |A| = |B| \]

\[ \Rightarrow A = \pm B \]

\[ \Rightarrow y - 1 = \pm \frac{1}{2} (x + 3) \]

\[ (1) \quad y - 1 = + \frac{1}{2} (x + 3) \]

\[ (2) \quad y - 1 = - \frac{1}{2} (x + 3) \]

But \( y(-1) = 0 \) (i.e. when \( x = -1, y = 0 \))

\[ \Rightarrow (1) \text{ is not possible} \]

At \( (x, y) = (-1, 0) \), \[
\begin{align*}
\text{LHS of (1)} &= -1 \\
\text{RHS of (1)} &= \frac{1}{2}(-1+3) = 1
\end{align*}
\]

And \[ \text{LHS of (2)} = -1 \]

\[ \text{RHS of (2)} = -\frac{1}{2}(-1+3) = -1 \]

\[ \Rightarrow (2) \text{ satisfies the initial condition} \]

Hence only (2) is the solution.
\[ y - 1 = -\frac{1}{2} (x + 3) \]

or \[ y = -\frac{1}{2} x - \frac{1}{2} \]