Lecture 12

§ 4.7 Variable-coefficient Equations

as opposed to “constant coefficient eqns"

\[ ay'' + by' + cy = f(x) \rightarrow \text{“constant coefficient”} \]

E.g. 1: \[ 2y'' + 2y' + y = f(x) \]

The coefficients are constants.

\[ a(x)y'' + b(x)y' + c(x)y = f(x) \rightarrow \text{“variable coefficient”} \]

E.g. 2: \[ x^2y'' - 4xy' + xy = f(x) \]

The coefficients are functions of \( x \).

First we introduce two theorems about two special types of variable-coefficient equations.
Thm 1: Consider \( y' + p(x)y = f(x) \)

If \( p, f \) are continuous on an interval \( I = (a,b) \) that contains \( x_0 \), then for any initial value \( y_0 \), there exists a unique solution \( y(x) \) defined on \( I \) to the I.V.P.

\[
\int y' + p(x)y = f(x) \quad y(x_0) = y_0.
\]
Thm 2. Consider \( y'' + p(x) y' + q(x) y = f(x) \)

If \( p, q, f \) are continuous on an interval \( I = (a, b) \) that contains \( x_0 \), then for any initial \( y_0, y_1 \), there exists a unique solution \( y(x) \) defined on \( I \) to the I.V.P.

\[
\begin{align*}
\{ & y'' + p(x) y' + q(x) y = f(x) \\
 & y(x_0) = y_0, \quad y'(x_0) = y_1
\end{align*}
\]

E.g. Does the I.V.P.

\[
\begin{align*}
\{ & x^2 y'' + xy' + y = e^x \\
 & y(1) = 1, \quad y'(1) = \sqrt{2}
\end{align*}
\]

have a unique solution \( y(x) \) on \( I = (0, +\infty) \)?

Divide the P.D.E. by \( x^2 \Rightarrow \text{Yes!} \)

\[
\begin{align*}
y'' + \frac{1}{x} y' + \frac{1}{x^2} y &= \frac{e^x}{x^2} \\
p(x) &= \frac{1}{x} \\
q(x) &= \frac{1}{x^2} \\
f(x) &= \frac{e^x}{x^2}
\end{align*}
\]

Are \( p, q, f \) continuous on \( I = (0, +\infty) \)?
only bad at $x=0 \& (0, +\infty)$

Note. In general, some I.V.P might not have a unique soln. Check the following Ex

**Ex** Consider \[ y' = 3y^{\frac{2}{3}} \quad (1) \]

**I.V.P** \[ y(2) = 0 \]

The I.V.P had two distinct solns!

① Check $y = 0$ is a soln to the I.V.P

② Check $y = (x-2)^3$ is also a soln to the I.V.P

A: ① Check $y = 0$:

LHS of (1) = $(0)' = 0$

RHS of (1) = $3 \cdot (0)^{\frac{2}{3}} = 0$

$\Rightarrow$ LHS = RHS

Moreover, $y(2) = 0$ is satisfied!

why? “$y=0$” means “$y$ equals to 0 at all points”
(2) Check $y = (x-2)^3$

$LHS$ of (1) = $((x-2)^3)' = 3(x-2)^2$

$RHS$ of (1) = $3y^{2/3} = 3((x-2)^3)^{2/3}$

$= 3(x-2)^2$

$\Rightarrow LHS = RHS$

Moreover, $y_{12} = (2-2)^3 = 0$ !

Q: Can we find the two solns by ourselves in the above?

A: Yes. This is a separable 1st order D.E

$$\frac{dy}{dx} = 1 - 3y^{2/3}$$

$f(x) g(y)$

Step 1: Check whether $g(y) = 0$ gives a soln.

$g(y) = 0 \Rightarrow y^{2/3} = 0 \Rightarrow y = 0$

It is a soln, as we checked.

Step 2: Now assume $g(y) \neq 0$. Then separate the variables $\Rightarrow$
\[
\frac{1}{3} \frac{dy}{y^{2/3}} = dx
\]
\[
\Rightarrow \frac{1}{3} \int y^{-2/3} \, dy = \int dx
\]
\[
\Rightarrow y^{1/3} = x + C
\]
\[
\Rightarrow y = (x+C)^3
\]
\[
y(2) = 0 \Rightarrow 0 = (2+C)^3 \Rightarrow C = -2
\]

Hence \( y = (x - 2)^3 \).
Next we discuss how to solve a special class of variable-coefficient eqns.

Recall in the end of lecture 11, we did the following

Eg:

Verify \( y_1 = x, \ y_2 = x^2 \) are solutions to the DE

\[ x^2 y'' - 4xy' + 4y = 0 \]

The above DE belongs to the class of so-called "Cauchy-Euler eqn".

Def: Cauchy-Euler eqn means the following 2nd order DE

\[ ax^2 y'' + bxy' + cy = f(x) \]

where \( a, b, c \in \mathbb{R} \).

Remark:

when we consider Cauchy-Euler eqn, most times we take \( x > 0 \).

Eg: "\( 3x^2 y'' + 11xy' - 3y = 0 \)"

is a Cauchy-Euler eqn.
Q: How to find solns to

$$a x^2 y'' + b x y' + c y = f(x), \quad x > 0$$

A:

First consider the case $f = 0 \Rightarrow

$$a x^2 y'' + b x y' + c y = 0 \quad (2)$$

Hint: try the test function $y = x^r$, $r$ to be determined.

Plug in $y = x^r$ (note $y' = r x^{r-1}$, $y'' = r(r-1) x^{r-2}$)

\[ \Rightarrow \text{LHS of (2)} = a x^2 r(r-1) x^r + b x \cdot r x^{r-1} + c \cdot x^r \]

\[ = a r(r-1) x^r + b r x^r + c x^r \]

\[ = (a r^2 + (b-a) r + c) x^r \]

How to make it $= \text{RHS of (2)} = 0$ ?

Need $a r^2 + (b-a) r + c = 0$

It is called the associated characteristic eqn of the D.E.
E.g.: Find particular soln(s) to

$$3x^2y'' + 11xy' - 3y = 0 \quad \text{for} \quad x > 0.$$  

$$\Rightarrow a = 3, \quad b = 11, \quad c = -3$$

Use the above idea: try \( y_p = x^r \)

$$\Rightarrow \text{we need to solve}$$

$$"ar^2 + (b-a)r + c = 0"$$

which is $$3r^2 + 8r - 3 = 0$$

$$\Rightarrow (3r-1)(r+3) = 0$$

$$\Rightarrow r_1 = \frac{1}{3}, \quad r_2 = -3 \quad \text{two distinct roots}$$

Hence we get two particular solns:

$$y_1 = x^{\frac{1}{3}}, \quad y_2 = x^{-3}$$

In the above E.g., "$$ar^2 + (b-a)r + c = 0"$$ has two distinct roots. What if it has only one repeated root or has complex roots \( \alpha \pm \beta i \)?

Here is the general algorithm:
Algorithm: find solns to
\[ ax^2 y'' + bxy' + cy = 0 \text{ for } x > 0 \] \hspace{1cm} (\star)

Step 1. \hspace{1cm} \text{Solve the quadratic eqn}
\[ ar^2 + (b-a)r + c = 0 \] \hspace{1cm} (3)

Step 2. \hspace{1cm} \begin{align*}
\text{Case (I)} & \quad \Delta > 0 \quad \text{If (3) has two distinct real roots } r_1, r_2, \\
& \quad \text{then we have two linearly independent solns:} \\
& \quad \hspace{1cm} y_1 = x^{r_1}, \quad y_2 = x^{r_2} \\
\text{Case (II)} & \quad \Delta = 0 \quad \text{If (3) has only one repeated root } r_0, \\
& \quad \text{then we have two linearly independent solns:} \\
& \quad \hspace{1cm} y_1 = x^{r_0}, \quad y_2 = x^{r_0} \ln x \\
\text{Case (III)} & \quad \Delta < 0 \quad \text{If (3) has complex roots } \alpha \pm \beta i, \\
& \quad \beta \neq 0, \text{ then we have two linearly independent solns:} \\
& \quad \hspace{1cm} y_1 = x^{\alpha} \cos(\beta \ln x), \quad y_2 = x^{\alpha} \sin(\beta \ln x)
\end{align*}
In any of the cases (I), (II), (III), the general soln to (*) is

\[ y = C_1 y_1 + C_2 y_2, \, C_1, C_2 \in \mathbb{R} \]

We will skip the proof for the above algorithm.

But we briefly discuss case (II).

Q: In Case (II), why \( y_2 = x^r \ln x \) is a soln?

A: In case (II), \( r_0 \) is the repeated root of

\[ \frac{A}{x^2} + \frac{B}{x} + \frac{C}{1} = 0 \]

We thus have

\[ B^2 - 4AC = \Delta = (b-a)^2 - 4ac = 0 \]

and by quadratic formula,

\[ r_0 = \frac{-(b-a) \pm \sqrt{\Delta}}{2a} = \frac{b-a}{-2a} \]

\[ \Rightarrow -2a r_0 = b-a \Rightarrow 2a r_0 + b-a = 0 \]
We next plug in \( y_2 = x^{r_0} \ln x \) to
\[
ax^2y'' + bxy' + cy = 0
\]

Note \( y_2' = r_0 x^{r_0-1} \ln x + x^{r_0-1} \)

\[
y_2'' = r_0(r_0-1)x^{r_0-2} \ln x + (2r_0-1)x^{r_0-2}
\]

\[
a x^2y_2'' + bxy_2' + cy_2
\]
\[
= a r_0(r_0-1)x^{r_0} \ln x + a(2r_0-1)x^{r_0} + br_0 x^{r_0} \ln x
\]
\[
+ bx^{r_0} + cx^{r_0} \ln x
\]
\[
= (ar_0^2 + (b-a)r_0 + c)x^{r_0} \ln x + (2ar_0 + b-a)x^{r_0}
\]
\[
= 0
\]

Now by the above algorithm, we can solve
\[
ax^2y'' + bxy' + cy = 0 \quad (*)
\]

Can we solve the general eqn:
\[
ax^2y'' + bxy' + cy = f(x) \quad (4)
\]
Yes! This was indeed already done by Lecture 11, variation of parameters.

Suppose \( y_1, y_2 \) are two linearly independent solns. to \((x)\), then we have a particular soln to \((4)\):

\[
y_p = v_1 y_1 + v_2 y_2
\]

\[
v_1 = \int \frac{f y_2}{a(x) W(y_1, y_2)} dx, \quad v_2 = \int \frac{f y_1}{a(x) W(y_1, y_2)} dx
\]

E.g. Find the general soln to

\[
x^2 y'' + 5xy' + 5y = 0 \quad \text{for } x > 0
\]

\[
a = 1 \quad b = 5 \quad c = 5
\]

Step 1: Solve \( a r^2 + (b-a) r + c = 0 \)

\[
\Rightarrow r^2 + 4r + 5 = 0
\]
\[ r = \frac{-4 \pm \sqrt{4^2-4(5)}}{2 \cdot 1} = -2 \pm i \]

Hence we have complex roots:

\[ y_1 = x^{-2} \cos(\ln x), \quad y_2 = x^{-2} \sin(\ln x) \]

General soln:

\[ c_1 y_1 + c_2 y_2 = c_1 x^{-2} \cos(\ln x) + c_2 x^{-2} \sin(\ln x) \]

E.g.: Solve the I.V.P

we can assume that \( x \) is close to \( x_0 = 1 \).

\[ y(1) = 1, \quad y'(1) = 2 \]

Thus \( x > 0 \)

A: Note \( y'' + \frac{1}{x} y' = 0 \) is not a Cauchy-Euler eqn.

\[ a x^2 y'' + b x y' + c y = 0 \]

But we can multiply it by \( x^2 \): \[ x^2 y'' + x y' = 0 \]

\[ \alpha = 1, \quad b = 1, \quad c = 0 \]
Step 1: Solve \( ar^2 + (b-a)r + c = 0 \)
\[ \Rightarrow r^2 + 0 = 0 \]
or \( r^2 = 0 \)
\[ \Rightarrow r_0 = 0, \text{ repeated root} \]

Step 2: Since we have only one repeated root: \( r_0 = 0 \)
\[ y_1 = x^{r_0} = x^0 = 1 \]
\[ y_2 = x^{r_0} \ln x = \ln x \]
And the general soln
\[ y = c_1 y_1 + c_2 y_2 \]
\[ = c_1 + c_2 \ln x \]

Step 3: plug in \( y(1) = 1, \ y'(1) = 2 \)
\[ x = 1, \ y = 1 \]
\[ y(1) = 1 \Rightarrow 1 = c_1 + c_2 \ln 1 \]
\[ \Rightarrow C_1 = 1 \]

To use \( y'(1) = 2 \), note
\[ y' = C_2 \frac{1}{x} \]

\[ y'(1) = 2 \Rightarrow 2 = C_2 \cdot 1 \]
\[ x = 1 \quad y = 2 \Rightarrow C_2 = 2 \]

\[ \Rightarrow y = 1 + 2 \ln x \]

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E.g. Find a particular soln to
\[ x^2 y'' + xy' = 1 \quad \text{for} \ x > 0 \quad (5) \]

\[ f(x) \]

A = step 1: Find two L.I solns to
the homogeneous eqn:
\[ x^2 y'' + xy' = 0 \]

We already did in the above. \( y_1 = 1; \ y_2 = \ln x \)
Step 2: Use variation of parameters to find a particular solution to (15):

\[ y_p = v_1 y_1 + v_2 y_2 \]

Here

\[ v_1 = \int \frac{-f y_2}{a(x) w(y_1, y_2)} \, dx \]

\[ v_2 = \int \frac{f y_1}{a(x) w(y_1, y_2)} \, dx \]

and

\[ w(y_1, y_2) = y_1 y_2' - y_2 y_1' = 1 (\ln x)' - \ln x (1)' = \frac{1}{x} \]

\[ \Rightarrow v_1 = \int \frac{-f y_2}{a(x) \frac{\ln x}{x^2} \frac{1}{x}} \, dx = \int \frac{-\ln x}{x} \, dx \]

\[ u = \ln x \]

Choose \( v_1 = -\frac{1}{2} (\ln x)^2 + C \)
\[ v_2 = \int \frac{1}{x^2} \cdot \frac{1}{x} \, dx = \int \frac{1}{x} \, dx = \ln x + C \]

Choose \( v_2 = \ln x \)

\[ \Rightarrow \quad y_p = v_1 y_1 + v_2 y_2 \]

\[ = - \frac{1}{2} (\ln x)^2 \cdot 1 + (\ln x) \cdot (\ln x) \]

\[ = \frac{1}{2} (\ln x)^2 \]