Lecture 10

Plan: § 4.5 The superposition principle

Consider: \( ay'' + by' + cy = f(x) \)

Last time: find "a particular soln"
  "some soln"

Today: find "all" the solns to the above D.E

Thm 1. (Superposition Principle)

Let \( y_{p,1} \) be a particular solution to the D.E

\[ ay'' + by' + cy = f_1(x) \]

and \( y_{p,2} \) be a particular solution to the D.E

\[ ay'' + by' + cy = f_2(x) \]

Then for any constants \( k_1 \) and \( k_2 \), the function \( k_1 y_{p,1} + k_2 y_{p,2} \) is a particular soln to the D.E.
\[ ay'' + by' + cy = k_1 f_1(x) + k_2 f_2(x) \]

**E.g.:**

1. Find a particular soln to

\[ y'' - y = 1 \]

2. Find a particular soln to

\[ y'' - y = x^2 \]

3. Find a particular soln to

\[ y'' - y = 2 - x^2 \]

**A:**

1. \[ \text{RHS} = 1 = 1 \cdot x^0 \cdot e^0 \cdot x \]

\[ \Rightarrow \sum_{m=0}^{\infty} r=0 \]
Consider \( ay'' + by' + cy = C_0 x^m e^{rx} \)

(I) If \( r \) is not a root of the char. eqn

\[ a\lambda^2 + b\lambda + c = 0 \]

then use the test function

\[ y = (A_m x^m + A_{m-1} x^{m-1} + \ldots + A_1 x + A_0) e^{rx} \]

Note the char. eqn is \( \lambda^2 - 1 = 0 \)

\( r = 0 \) is not a root of the char. eqn.
Thus we use the test function

\[ y = A_0 e^{0 \cdot x} = A_0 \]

Then \( \text{LHS} = y'' - y = -A_0 \)

Note \( \text{RHS} = 1 \)

\( \Rightarrow \) we need \(-A_0 = 1\)

\( \Rightarrow \) \( A_0 = -1 \)

Hence \( y_{p,1} = -1 \) is a particular solution to

\[ y'' - y = 1 \]
(2) RHS = $x^2 = 1 \cdot x^2 \cdot e^{0 \cdot x}$

$\Rightarrow \begin{cases} m = 2 \\ r = 0 \end{cases}$

Again note the char. eqn is

$\lambda^2 - 1 = 0$

and $r = 0$ is NOT a root.

$\Rightarrow$ we use the test function $y = (A_2 x^2 + A_1 x + A_0) e^{0 \cdot x}$

plug into $y'' - y = x^2$

$\Rightarrow LHS \begin{array}{c} \quad 2A_2 - (A_2 x^2 + A_1 x + A_0) = X^2 \\ \overbrace{y''}^{y} \end{array}$

$\Rightarrow -A_2 x^2 - A_1 x + (2A_2 - A_0) = X^2$

Compare the two sides:
\( x^2 \text{- term: } -A_2 x^2 = x^2 \)

\( x \text{- term: } -A_1 x = 0 \cdot x \)

\( \text{Constant-term: } 2A_2 - A_0 = 0 \)

\[ \Rightarrow \begin{cases} -A_2 = 1 \\ -A_1 = 0 \\ 2A_2 - A_0 = 0 \end{cases} \Rightarrow \begin{cases} A_2 = -1 \\ A_1 = 0 \\ A_0 = -2 \end{cases} \]

\[ \Rightarrow y_{p,2} = -x^2 - 2 \text{ is a particular soln to } y'' - y = x^2 \]

By (1), \( y_{p,1} = -1 \) is a particular soln to \( y'' - y = 1 \)

By (2), \( y_{p,2} = -x^2 - 2 \) is a particular soln to \( y'' - y = x^2 \)

By Thm 1,
"$k_1y_{p,1} + k_2y_{p,2}$ is a particular soln to $y'' - y = k_1 \cdot 1 + k_2 \cdot x^2$

In particular, let $k_1 = 2$, $k_2 = -1$

$$\Rightarrow \quad y = \left[ 2y_{p,1} - y_{p,2} \right] = 2 \cdot (-1) - (-x^2 - 2) = x^2$$

is a particular soln to

$$y'' - y = 2 - x^2$$

A very often used case of Thm 1 is $k_1 = 1$ and $k_2 = 1$. That is:

Corollary:
If $y_{p,1}$ is a particular soln to

$$ay'' + by' + cy = f_1(x)$$

and $y_{p,2}$ is a particular soln to
\[ ay'' + by' + cy = f_2(x) \]

then \( y_{p,1} + y_{p,2} \) is a particular soln to
\[ ay'' + by' + cy = f_1(x) + f_2(x) \].

Q: What does the above Corollary say if \( f_1 = f \), \( f_2 = 0 \).

A: Assume \( y_p \) is a particular soln to
\[ ay'' + by' + cy = f(x) \]. (1)

Recall we know how to find general solns to
\[ ay'' + by' + cy = 0 \] (2)

(\text{Assume } y_1, y_2 \text{ are two linearly independent solns to (2). Then the general solns to (2): } C_1 y_1 + C_2 y_2)
Then \( y_p + (C_1y_1 + C_2y_2) \) is also a soln to
\[
ay'' + by' + cy = f(x) + 0 = f(x) \tag{1}
\]

Q: Are there any other solns to (1)?
A: No! \( y_p + C_1y_1 + C_2y_2 \) gives all possible solns to (1)!

Thm: Suppose \( y_p \) is a particular soln to
\[
ay'' + by' + cy = f(x). \tag{1}
\]
Suppose \( y_1, y_2 \) are two linearly independent solns to
\[
ay'' + by' + cy = 0. \tag{2}
\]
Then the general solutions to (1) are (meaning “all solutions”)

\[ y_p + C_1 y_1 + C_2 y_2, \quad C_1, C_2 \in \mathbb{R} \]

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Idea of proof:

Let \( y_0 \) be any other solution to (1) (than \( y_p \)). \[ ay_0 + by_0' + cy_0 = f(x) \]

We want to show:

\[ y_0 = y_p + C_1 y_1 + C_2 y_2 \]

for some \( C_1, C_2 \in \mathbb{R} \).

Claim: \( (y_0 - y_p) \) is a solution to (2).

Why? Check \( y_0 - y_p \) satisfies (2).
\[ a(y_0 - y_p)'' + b(y_0 - y_p)' + c(y_0 - y_p) \]

\[ = \left( ay_0'' + by_0' + cy_0 \right) - \left( ay_p'' + by_p' + cy_p \right) \]

\[ = 0 = \text{RHS of (2)} \]

Hence \( y_0 - y_p \) is a soln of (2).

But we know every soln of (2) can be written as \( c_1 y_1 + c_2 y_2 \), \( c_1, c_2 \in \mathbb{R} \)

\[ \Rightarrow \quad y_0 - y_p = c_1 y_1 + c_2 y_2 \]

\[ \Rightarrow \quad y_o = y_p + c_1 y_1 + c_2 y_2 \]

Defn: \( "ay'' + by' + cy = 0" \) is called the associated homogeneous D.E of \( "ay'' + by' + cy = f(x)" \).
Algorithm to find the general solns (that is, all solns) to
\[ ay'' + by' + cy = f(x) \] (1)

Step 1: Find the general solns to the associated homogeneous D.E
\[ ay'' + by' + cy = 0 \]
Call the solns \( y_h = C_1y_1 + C_2y_2 \)

Step 2: Find a particular soln. to (1)
How? We can use Lecture 9 "undetermined coeff. method"
(next lecture will discuss another method)
Call the soln \( y_p \)
Step 3. Add up solutions in step 1, 2

\[ y_p + c_1 y_1 + c_2 y_2, \quad c_1, c_2 \in \mathbb{R} \]

E.g.: ① Find the general solutions to

\[ y'' - y = 2 - x^2 \quad (3) \]

② Solve the I.V.P

\[ \begin{cases} y'' - y = 2 - x^2 \\ y(0) = 1, \ y'(0) = 0 \end{cases} \]

A: ① We follow the algorithm:

Step 1: Solve the associated homogeneous D.E.

\[ y'' - y = 0 \]
The char. eqn. \( \lambda^2 - 1 = 0 \)

\( \Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = -1 \)

Hence \( y_h = C_1 e^x + C_2 e^{-x} \)

Step 2: Find a part. soln. to (3).

Already did this early today!

\( y_p = x^2 \)

Step 3: Add them up

\[ y = y_p + y_h \]

\[ = x^2 + C_1 e^x + C_2 e^{-x} \]

This is the general solns to (3)
Recall the general solutions to (3):

\[ y = x^2 + C_1 e^x + C_2 e^{-x} \]

\[ y(0) = 1 \implies 1 = 0^2 + C_1 e^0 + C_2 e^{-0} \]

\[ x = 0, y = 1 \implies C_1 + C_2 = 1 \quad \text{(1)} \]

Note \( y' = 2x + C_1 e^x - C_2 e^{-x} \)

\[ y'(0) = 0 \implies 0 = 2 \cdot 0 + C_1 e^0 - C_2 e^{-0} \]

\[ x = 0, y' = 0 \implies C_1 - C_2 = 0 \quad \text{(2)} \]

\[ \implies \begin{cases} C_1 = \frac{1}{2} \\ C_2 = \frac{1}{2} \end{cases} \]

The solution to I.V.P is

\[ y = x^2 + \frac{1}{2} e^x + \frac{1}{2} e^{-x} \]