Separable Differential Equations

- Formula
  \[ y' = \frac{a(x)}{b(y)} \]

- Method of Solution
  - First check if \( y' = 0 \), solve for \( y \) and see if this equation is a solution
  - Then we can rearrange the equation to: \( b(y)dy = a(x)dx \) and integrate

- Example
  - Population \( P(t) \) can be described as \( P' = (k \cos t)P \). Find the solution if the initial population is 100, and \( k = 5 \)
    \[ \int \frac{1}{P}dP = \int (k \cos t)dt \]
    \[ P = Ce^{k \sin t} \]
    \[ P = 100e^{5 \sin t} \]

Linear First-Order Equations

- Formula
  \[ y' + P(x)y = Q(x) \]

- Method of Solution
  - Calculate integrating factor \( \mu(x) = e^{\int P(x)dx} \)
  - \[ y = \frac{\int \mu(x)Q(x)dx + C}{\mu(x)} \]

- Example
  - Solve \( y' + 3x^2y = x^2 \) with initial condition \( y(0) = 2 \).
    \[ \mu(x) = e^{x^3} \]
    \[ y = \frac{\int e^{x^3}x^2dx}{e^{x^3}} \rightarrow u = x^3 \rightarrow du/3 = x^2dx \rightarrow y = \frac{\int e^udu}{3e^{x^3}} = \frac{1}{3} + \frac{c}{3e^{x^3}} \]
    \[ -2 = \frac{1}{3} + \frac{c}{3e^0} = \frac{1}{3} + \frac{c}{3} \rightarrow c = 5 \]

Exact Equations

- Formula
  \[ M(x, y) + N(x, y)\frac{dy}{dx} = 0 \]

- We must either show \( \int M(x, y)dx = \int N(x, y)dy = F(x, y) \) or
\[ \frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} \]

- **Method of Solution**
  - Integrate \( M \) and \( N \) in terms of \( x \) and \( y \), respectively, and solve for \( F \).
  - \( F(x, y) + c = 0 \) is an implicit solution

- **Example**
  - \( M = (x + y)^2; \quad N = (2xy + x^2 + y^2) \)
  - \( \int M \, dx = \frac{(x + y)^3}{3} + g(y) = \frac{x^3}{3} + xy^2 + \frac{y^3}{3} + g(y) \)
  - \( \int N \, dy = xy^2 + x^2y + \frac{y^3}{3} + h(x) \)
  - \( h(x) = \frac{x^3}{3} + c, \quad g(y) = c \)
  - \( F(x, y) = \frac{(x + y)^3}{3} + c = 0 \)
  - \( c = -\frac{8}{3} \)

- **Special Integrating Factors**
  - **Formula:**
    - \( M(x, y) + N(x, y) y' = 0 \)
    - We must either show:
      - *Case 1:* \( \frac{M_y - N_x}{N} \) is solely a function of \( x \), or
      - *Case 2:* \( \frac{N_x - M_y}{M} \) is solely a function of \( y \)
  - **Method of solution:**
    - Multiply both sides of equation by integrating factor \( \mu(x) = e^{\int \frac{M_y - N_x}{N} \, dx} \) (Case 1) or \( \mu(y) = e^{\int \frac{N_x - M_y}{M} \, dy} \) (Case 2)
    - The equation is now exact and we can solve it as described previously
  - **Example:**
    - Solve \( x + (x^2y + 4y)y' = 0, \quad y(4) = 0 \)
    - \( M = x, \quad N = x^2y + 4y \rightarrow M_y = 0, \quad N_x = 2xy \)
    - \( \frac{M_y - N_x}{N} = \frac{-2x}{(x^2 + 4)} \cdot \frac{N_x - M_y}{M} = 2y \)
    - Multiply by \( \mu(y) = e^{y^2} \) on both sides, so new equation is \( xe^{y^2} + (x^2y + 4y)e^{y^2} y' = 0 \)
    - \( M_{\text{exact}} = xe^{y^2}, \quad N_{\text{exact}} = (x^2 + 4)ye^{y^2} \)
\[
\int x e^{y^2} \, dx = \frac{x^2 e^{y^2}}{2} + g(y)
\]

\[
- \int (x^2 + 4) y e^{y^2} \, dy \rightarrow u = y^2 \rightarrow \frac{du}{2} = y \, dy \rightarrow \int (x^2 + 4) e^u \, du = (x^2 + 4) \frac{e^{y^2}}{2} + h(x)
\]

- \( g(y) = 2 e^{y^2} + c \), \( h(x) = c \)
- \( F(x, y) = \frac{x^2 e^{y^2}}{2} + 2 e^{y^2} + c = 0 \)
- \( c = -10 \)

**Homogeneous Linear Second-Order ODEs with Constant Coefficients**

- **Equation:**
  - Method of solution:
    - Assuming that this equation will have a solution of the form \( y = e^{mx} \)
    - Solve the *characteristic equation*: \( am^2 + bm + c = 0 \) using quadratic formula:
      \[
m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
    - Three cases for our solution:
      * Two real roots: general solution is \( y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \)
      * One real root: general solution is \( y = c_1 e^{mx} + c_2 x e^{mx} \)
      * Imaginary roots: general solution is \( y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x] \), where \( m = \alpha \pm \beta i \)

- **Example:**
  - Solve \( 3y'' + 2y' + y = 0 \), \( y(0) = 5 \), \( y'(0) = 1 \)
    - \( 3m^2 + 2m + 1 = 0 \rightarrow m = \frac{-1 \pm \sqrt{2}}{3} \)
    - \( y = e^{-x/3} \left[ c_1 \cos \left( \frac{\sqrt{2}}{3} x \right) + c_2 \sin \left( \frac{\sqrt{2}}{3} x \right) \right] \)
  - After taking derivative and substituting for initial conditions we get \( c_1 = 5 \), \( c_2 = 4 \sqrt{2} \)